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Wave functions and correlation functions for GKP strings from integrability

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Abstract

We develop a general method of computing the contribution of the vertex operators to the semi-classical correlation functions of heavy string states, based on the state-operator correspondence and the integrable structure of the system. Our method requires only the knowledge of the local behavior of the saddle point configuration around each vertex insertion point and can be applied to cases where the precise forms of the vertex operators are not known. As an important application, we compute the contributions of the vertex operators to the three-point functions of the large spin limit of the Gubser-Klebanov-Polyakov (GKP) strings in AdS_3 spacetime, left unevaluated in our previous work [arXiv:1110.3949] which initiated such a study. Combining with the finite part of the action already computed previously and with the newly evaluated divergent part of the action, we obtain finite three-point functions with the expected dependence of the target space boundary coordinates on the dilatation charge and the spin.

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1 Introduction

For the study of conformally invariant theories, understanding of two point and three point functions constitutes the crux of the matter. Two point functions encode the spectrum while three point functions determine the essential dynamics, and together they govern the entire theory, at least in principle. This of course applies to the study of AdS/CFT correspondence [1–3], in which the conformal invariance plays the key role.

In this article, we will be concerned with such correlation functions in the best studied duality of that type, namely the one between the $N = 4$ super Yang-Mills (SYM) theory in four dimensions and the type IIB superstring theory in $AdS_5 \times S^5$ spacetime. During the past fifteen years since the inception of such a duality, the scope of the studies has been expanded from the BPS to the non-BPS sectors and from the two-point to the three-point functions. The history and the achievements made through such studies are summarized, up to around 2010, in the comprehensive collection of review articles [4].

In the course of the development, the integrability found on both sides of the correspondence played a pivotal role. On the SYM side, the problem of finding the perturbative spectrum of the composite operators, equivalently their two point functions, has been mapped to the analysis of integrable spin chains and many precise results have been obtained. Such approach was subsequently generalized to the three point functions [5–7] and the new underlying integrable structure is now beginning to be uncovered [8–18].

On the other hand the existence of the classical integrability of the string sigma model in $AdS_5 \times S^5$ was proven in the early days of the development [19] and is later used to characterize various classical solutions [20–22]. However it was rather recently that people began to understand how such classical solutions can be utilized in the saddle point calculation of two point functions [23–25]. As for the study of three point functions, it is substantially more difficult, mainly because the relevant saddle point solutions are in general hard to construct. One special class for which this difficulty can be avoided is the so-called heavy-heavy-light (HHL) correlators [26–28]. In this case, since the light vertex operator, carrying a small quantum number, does not affect the saddle configuration determined by the other two heavy operators, one can treat it as a "perturbation" of the two point function with relative ease. In this way results for a variety of HHL correlators have been obtained [29–39]. Computations have also been made of a more general yet still restricted class of three point functions, where one can approximate the AdS part of the saddle point configuration by that of the zero mode of the string [40–42]. This should be valid for BPS or near BPS strings with charges in the S^5 part. For other studies on

the three point functions from different viewpoints, see [43–49].

Studies of the three point functions of genuine non-BPS heavy states have been initiated recently in [50] for the strings rotating in S^k but not in the AdS space, and in our previous paper [51] for the large spin limit of the GKP strings [52] (to be called LSGKP strings) spinning in AdS_3 but sitting at a point on a sphere. In these works, the classical integrability of the system is fully exploited, as in the computation of the gluon scattering amplitudes [53–57], and the essential information can be extracted without the explicit knowledge of the saddle point configuration. In this way, the universal contribution from the AdS_2 part was obtained in [50], while the essential part of the three point coupling for the LSGKP strings, including the dependence on the spins of the strings, has been computed in our previous work. However, in both of these works, the complete answers for the three point functions were not yet obtained. For [50], the contribution from the motion in the S^k part was not computed. On the other hand, in our previous work, the contributions of the divergent part of the action and of the vertex operators, which together should give a finite result including the dependence on the vertex insertion positions on the boundary, were left unevaluated. Such computations remain as the most urgent problems to be solved in this line of approach.

In the present work, we shall give a solution to the problem left unanswered in our previous study. Namely, we develop a powerful method of computing the contribution of the vertex operators, which can be applied to the cases where both the precise form of the vertex operators and the exact form of the the saddle point solution are not available. Using this method, we will be able to complete the computation of the three point functions for the LSGKP strings started in our previous work. What we will need are only the local asymptotic behaviors of the solution near the vertex insertion points and certain global symmetry transformations. Then the integrability and the analyticity possessed by the system will allow us to unite such informations and compute the desired contributions.

Let us be more specific and describe the essence of our method. The structure of the three point function in the semi-classical approximation can be expressed schematically in the following way:

$$G(x_1, x_2, x_3) = e^{-S[X_*]_{\text{finite}}} \left[e^{-S[X_*]_\epsilon} \prod_{i=1}^{3} V_i[X_*; x_i, Q_i]_\epsilon \right] . \tag{1.1}$$

It consists of the contribution of the action and that of the vertex operators, evaluated on the saddle point configuration denoted by X_*. The contribution of the action can be split, in an appropriate way, into the finite part $S[X_*]_{\text{finite}}$ and the divergent part $S[X_*]_\epsilon$. $V_i[X_*; x_i, Q_i]_\epsilon$ denotes the value of the i-th vertex operator on X_*, which carries

a large charge $Q_i \sim O(\sqrt{\lambda})$, where λ is the 't Hooft coupling, and is located at x_i on the boundary of the AdS space. The subscript ϵ on $S[X_*]_\epsilon$ and the vertex operators signifies that these quantities contain divergences which are regularized by a small parameter ϵ. As it will be reviewed in section 2, with the use of the Pohlmeyer reduction and a newly developed generalized Riemann bilinear identity, we were able to compute in [51] the quantity $S[X_*]_{\text{finite}}$, which was called the "regularized area". But unfortunately the rest of the contributions shown in the square bracket in (1.1), in particular the one from the vertex operators, was not obtained.

There are two major difficulties in computing the contribution of the vertex operators. One is that even semi-classically the precise form of the conformally invariant vertex operator for the LSGKP string is not known. The other is the difficulty to construct the saddle point solution for the three point function. We suggested in our previous work that the appropriate way to compute the contribution of the vertex operators would be to make use of the semi-classical wave functions, which are related to the vertex operators by the state-operator correspondence. In particular, if one can find the action-angle variables of the system, such wave functions can be constructed easily since the Hamilton-Jacobi equations become simple. But of course finding the action-angle variables is a difficult task for non-linear systems and it is even more difficult to evaluate the wave functions constructed in terms of them without knowing the saddle point solution. For these reasons we could not implement the idea above explicitly for the GKP strings.

We now describe two essential observations which allow us to overcome these difficulties.

The first observation is that for the class of so-called finite gap solutions, to which the GKP solution belongs, one can construct the action-angle variables by the method of "separation of variables" developed by Sklyanin [58]. For the case of the string in AdS_3, due to the existence of the Virasoro constraints, the analytic structures of the relevant quasi-momentum and the spectral curve are modified from those for the well-studied case of the string in $AdS_3 \times S^1$ [59–61], and one has to analyze them carefully. Nonetheless, we can show that the Sklyanin's method is still applicable. The details of this construction will be given in section 3.2.

The second important observation is that the values of the angle variables, which determine the value of the wave function corresponding to a vertex operator, can be computed from the behavior of the local solution near the position of the vertex operator. This is intimately related to the recognition that the choice of the normalization of the so-called Baker-Akhiezer eigenvector, which governs the value of the angle variables in

the Sklyanin's method, can be characterized by the global symmetry transformation that generates the local solution around the vertex operator from a common reference solution. More precise description will be given in section 3.3.

Based on these observations, we develop a general method which leads to explicit procedures and formulas for computing the contributions of the vertex operators, which generate string configurations described by finite gap solutions. To check the validity and the power of the method, we first apply it to the computation of two point functions. In this case, our method is particularly simple and makes clear that essentially it constructs the representation of the global conformal symmetry in terms of the boundary coordinates through combination of the wave functions. Due to this general feature, the detailed form of the saddle point solution is not important and we can compute the two point functions for the general elliptic GKP strings, *i.e.* without taking the large spin limit.

Next we apply the method to the main subject of our study, the three point functions of the LSGKP strings. The basic procedure is the same around each vertex operator. These contributions from the vertex operators can be reexpressed in terms of the differences of the positions of the vertex operators on the boundary, producing the correct representation of the conformal symmetry on the three point function for states carrying the spin as well as the conformal dimension. One important difference from the case of the two point function is that, together with such expressions made up of the combinations of position coordinates, there appear additional factors which carry the information about how the three states intertwine. Remarkably, they can be expressed in terms of the contour integrals around the three singularities on the worldsheet, which played crucial roles in the computation of the finite part of the action performed in our previous work. Consequently, we can evaluate such quantities in a similar fashion as before. Moreover, the divergent part of the action, which is the last remaining piece to be computed, can also be represented by the same type of contour integrals through the Riemann bilinear identity and this makes the mechanism of the cancellation of divergences transparent.

The final result for the three point function for the LSGKP strings is given in (5.55). It exhibits the correct dependence on the boundary coordinates expected of the three point functions carrying the spin and reduces, in the appropriate limit, to the properly normalized two point function. Unfortunately, comparison with the corresponding correlator on the SYM side is not possible as it is not available at present time.

The organization of the rest of the article is as follows. To make this article more or less self-contained, we will give, in section 2, a brief review of the method and the result of our previous work and set the notations. In section 3, we develop a general method

for evaluating the contributions of the vertex operators. Since this section contains a variety of materials, we will first sketch the general strategies in subsection 3.1. We then describe in subsection 3.2 how one can construct the action-angle variables for a string in the (Euclidean) AdS_3 by applying the Sklyanin's method. In order to discuss the generic structure of the action-angle variables, we make the analysis from the universal point of view of "infinite gap" solutions. Section 3.3 explains how one can evaluate the angle variables and the wave functions by the use of appropriate global symmetry transformations. The general method developed up to this point is then applied to the computation of the two point functions in section 4. Namely, we derive the general formula in section 4.1 and apply it to the case of the GKP string in section 4.2. Finally, in section 5 we will complete the calculation of the three point functions for the LSGKP string initiated in our previous paper. In section 5.1, by using the generalized Riemann bilinear identity, we express the divergent part of the action in such a form that its cancellation with the divergent part of the wave functions is easy to understand. Then in section 5.2 the contributions of the wave functions are evaluated by applying our general method. These results as well as the finite part of the action computed in our previous paper are put together in section 5.3 to give the final form of the three point function. We conclude and indicate future directions in section 6 and provide a number of appendices to give some useful details.

2 A brief review of the previous work

We begin by giving a concise review of the methods and the results obtained in our previous work and set the notations.

The LSGKP strings of our interest live in the AdS_5 spacetime with the embedding coordinates $(X_{-1}, X_0, X_1, X_2, X_3, X_4)$ satisfying $-X_{-1}^2 - X_0^2 + X_1^2 + \cdots + X_4^2 = -1$. They are related to the Poincaré coordinates $(x^\mu, z) = (x^0, x^1, x^2, x^3, z)$ as

$$X_{-1} + X_4 = \frac{1}{z}\,, \qquad X_{-1} - X_4 = z + \frac{x^\mu x_\mu}{z}\,, \qquad X_\mu = \frac{x_\mu}{z}\,, \tag{2.1}$$

where $z = 0$ corresponds to the boundary of AdS_5. In what follows, we will focus on the solutions which propagate in the subspace spanned by $\vec{X} \equiv (X_{-1}, X_1, X_2, X_4)$, having the structure of Euclidean AdS_3. The action on the Euclidean worldsheet parametrized by the plane coordinate $(z, \bar{z})$ is proportional to the area A and takes the form

$$S = TA = 2T \int d^2z (\partial \vec{X} \cdot \bar{\partial} \vec{X} + \Lambda(\vec{X} \cdot \vec{X} + 1))\,, \tag{2.2}$$

where $T = \sqrt{\lambda}/2\pi$ is the string tension, $\vec{A} \cdot \vec{B} \equiv -A_{-1}B_{-1} + A_1B_1 + A_2B_2 + A_4B_4$, and Λ is the Lagrange multiplier. After eliminating Λ, the equation of motion for $\vec{X}$ is given by

$$\partial\bar{\partial}\vec{X} = (\partial\vec{X} \cdot \bar{\partial}\vec{X})\vec{X} \,. \tag{2.3}$$

In addition we must impose the Virasoro constraints

$$\partial\vec{X} \cdot \partial\vec{X} = \bar{\partial}\vec{X} \cdot \bar{\partial}\vec{X} = 0 \,. \tag{2.4}$$

The global isometry group for real $\vec{X}$ is $SO(1,3)$. However, as we shall be concerned with the solutions which are the saddle point configurations for the semi-classical correlation functions, we need to allow complex solutions and the global symmetry group of the system should be taken as $SO(4,C) \simeq SL(2,C)_L \times SL(2,C)_R$. To exhibit its action it is convenient to introduce the matrix with unit determinant

$$\mathbb{X} \equiv \begin{pmatrix} X_+ & X \\ \bar{X} & X_- \end{pmatrix} , \qquad \det \mathbb{X} = 1 \,, \tag{2.5}$$

where

$$X_+ = X_{-1} + X_4 \,, \qquad X_- = X_{-1} - X_4 \,, \tag{2.6}$$

$$X = X_1 + iX_2 \,, \qquad \bar{X} = X_1 - iX_2 \,. \tag{2.7}$$

Then $SL(2)_L \times SL(2)_R$ acts as

$$\mathbb{X}' = V_L \mathbb{X} V_R \,, \qquad V_L \in SL(2)_L \,, \quad V_R \in SL(2)_R \,. \tag{2.8}$$

A typical LSGKP solution, which we shall call "the reference solution", can be written as

$$\mathbb{X}^{\text{ref}} = \begin{pmatrix} e^{-\kappa\tau} \cosh \rho(\sigma) & e^{\kappa\tau} \sinh \rho(\sigma) \\ e^{-\kappa\tau} \sinh \rho(\sigma) & e^{\kappa\tau} \cosh \rho(\sigma) \end{pmatrix} , \tag{2.9}$$

where (τ, σ) are the Euclidean cylinder coordinates on the worldsheet and κ is a positive parameter. The function $\rho(\sigma)$, periodic with period 2π, is given in the interval $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ by

$$\rho(\sigma) = \begin{cases} \kappa\sigma \,, & \left(-\frac{\pi}{2} \le \sigma \le \frac{\pi}{2}\right) \\ \kappa\,(\pi - \sigma) \,, & \left(\frac{\pi}{2} \le \sigma \le \frac{3\pi}{2}\right) \end{cases} \,. \tag{2.10}$$

Note that this solution starts at $\tau = -\infty$ from the boundary and reaches the horizon at $\tau = \infty$. It will be useful to view the class of solutions of our interest, namely those which start from the boundary and end also on the boundary, as obtained from the standard solution above by applying appropriate $SL(2)_L \times SL(2)_R$ symmetry transformations. There

are two conserved global charges in this system, namely the dilatation charge Δ and the angular momentum (spin) S in the 1-2 plane. They are given in terms of the parameter κ by

$$\Delta = \frac{\sqrt{\lambda}}{2\pi}\kappa \int_0^{2\pi} d\sigma \, \cosh^2 \rho = \frac{\sqrt{\lambda}}{2\pi}(\kappa\pi + \sinh \kappa\pi)\,, \tag{2.11}$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\kappa \int_0^{2\pi} d\sigma \, \sinh^2 \rho = \frac{\sqrt{\lambda}}{2\pi}(-\kappa\pi + \sinh \kappa\pi)\,. \tag{2.12}$$

One of the major difficulties for computing the correlation functions of three or more LSGKP strings as the external states is that the relevant classical saddle point configurations have not been found. Fortunately, this obstacle can be overcome by making use of the classical integrability possessed by the system. The most convenient framework turned out to be the method of Pohlmeyer reduction[1], which proved extremely powerful in handling a similar situation in the analysis of the minimal surface problem encountered in the computation of the gluon scattering amplitudes at strong coupling [53–57]. In this method, one deals with the $SL(2)_L \times SL(2)_R$ invariant reduced degrees of freedom α, p and $\bar{p}$ defined by

$$e^{2\alpha} = \frac{1}{2}\partial\vec{X}\cdot\bar{\partial}\vec{X}\,, \quad p = \frac{1}{2}\vec{N}\cdot\partial^2\vec{X}\,, \quad \bar{p} = -\frac{1}{2}\vec{N}\cdot\bar{\partial}^2\vec{X}\,, \tag{2.13}$$

where $\vec{N}$ is a unit-normalized vector orthogonal to $\vec{X}, \partial\vec{X}$ and $\bar{\partial}\vec{X}$. The information of the original equations of motion as well as the Virasoro conditions can be encoded in the flatness of certain 2×2 $SL(2)$ connection matrices $B^L_z, B^L_{\bar{z}}, B^R_z, B^R_{\bar{z}}$, namely $\left[\partial + B^L_z, \bar{\partial} + B^L_{\bar{z}}\right] = 0$ and $\left[\partial + B^R_z, \bar{\partial} + B^R_{\bar{z}}\right] = 0$. These conditions lead to

$$\partial\bar{\partial}\alpha - e^{2\alpha} + p\bar{p}e^{-2\alpha} = 0\,, \tag{2.14}$$

$$\partial\bar{p} = \bar{\partial}p = 0\,. \tag{2.15}$$

Moreover, reflecting the integrability of the system, one can construct a flat connection with an arbitrary complex spectral parameter ξ out of these connections. Explicitly its components are given by

$$B_z(\xi) = \frac{1}{\xi}\Phi_z + A_z\,, \qquad B_{\bar{z}}(\xi) = \xi\Phi_{\bar{z}} + A_{\bar{z}}\,, \tag{2.16}$$

$$A_z \equiv \begin{pmatrix} \frac{1}{2}\partial\alpha & 0 \\ 0 & -\frac{1}{2}\partial\alpha \end{pmatrix}, \qquad A_{\bar{z}} \equiv \begin{pmatrix} -\frac{1}{2}\bar{\partial}\alpha & 0 \\ 0 & \frac{1}{2}\bar{\partial}\alpha \end{pmatrix}, \tag{2.17}$$

$$\Phi_z \equiv \begin{pmatrix} 0 & -e^{\alpha} \\ -pe^{-\alpha} & 0 \end{pmatrix}, \qquad \Phi_{\bar{z}} \equiv \begin{pmatrix} 0 & -\bar{p}e^{-\alpha} \\ -e^{\alpha} & 0 \end{pmatrix}. \tag{2.18}$$

[1]For the evaluation of the contribution from the vertex operators, which is the main focus of the present work, the finite gap integration method will also be indispensable .

The original connections are identified as those at the special values of the spectral parameter (with a similarity transformation for B^R) in the manner

$$B_z^L = B_z(\xi = 1)\,, \qquad B_{\bar{z}}^L = B_{\bar{z}}(\xi = 1)\,, \tag{2.19}$$

$$B_z^R = \mathcal{U}^\dagger B_z(\xi = i)\mathcal{U}\,, \qquad B_{\bar{z}}^R = \mathcal{U}^\dagger B_{\bar{z}}(\xi = i)\mathcal{U}\,, \tag{2.20}$$

$$\mathcal{U} = e^{i\pi/4}\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}. \tag{2.21}$$

Now the flatness of $B(\xi)$ is equivalent to the compatibility of the set of linear equations

$$(\partial + B_z(\xi))\psi(\xi) = 0\,, \qquad (\bar{\partial} + B_{\bar{z}}(\xi))\psi(\xi) = 0\,, \tag{2.22}$$

known as the auxiliary linear problem. Once the two independent solutions $\psi(\xi)$ of this system are obtained, one can immediately get the solutions for the left and right auxiliary linear problems involving the connections B^L and B^R as $\psi^L = \psi(\xi = 1)\,, \psi^R = \mathcal{U}^\dagger\psi(\xi = i)$. More specifically, we denote the solutions as $\psi^L_{\alpha,a}$ and $\psi^R_{\dot{\alpha},\dot{a}}$, where $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$ refer to the matrix indices of B^L and B^R (*i.e.* the $SL(2)_{L,R}$ spinor indices), while $a = 1, 2$ and $\dot{a} = 1, 2$ label the two independent solutions. What will be of great importance is the $SL(2)$-invariant product between two spinors given by

$$\langle\psi, \chi\rangle \equiv \epsilon^{\alpha\beta}\psi_\alpha\chi_\beta\,, \qquad (\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}\,, \quad \epsilon^{12} \equiv 1)\,. \tag{2.23}$$

One can then show that the solutions $\psi^{L,R}$ can be normalized as

$$\langle\psi^L_a, \psi^L_b\rangle = \epsilon_{ab}\,, \qquad \langle\psi^R_{\dot{a}}, \psi^R_{\dot{b}}\rangle = \epsilon_{\dot{a}\dot{b}}\,, \tag{2.24}$$

where ϵ_{ab} is the anti-symmetric tensor with $\epsilon_{12} \equiv 1$. With such normalized solutions, one can reconstruct the solution for the matrix $\mathbb{X}$ of the embedding coordinates as

$$\mathbb{X}_{a\dot{a}} = \psi^L_{1,a}\psi^R_{\dot{1},\dot{a}} + \psi^L_{2,a}\psi^R_{\dot{2},\dot{a}}\,. \tag{2.25}$$

In the discussion so far, the spectral parameter ξ has not played any significant role: It merely served as a convenient device to handle left and right problems in a unified manner. Moreover, the Pohlmeyer reduction by itself does not generate the solution of the non-linear equations of motion. (Afterall, the LSGKP solution is known from the beginning.) However, this formalism will be extremely useful for the computation of three (and higher) point functions for which the saddle point solutions are not known. The reason is as follows. It allows us to characterize the *local* behavior of the solution in the vicinity of the sources, *i.e.* the vertex operators, by certain functions of ξ. Moreover, through the analyticity property in ξ these local behaviors can be interconnected and yield such *global* information as the value of the area.

The direct $SL(2)$-invariant data characterizing the LSGKP solutions are the form of the functions $p(z), \bar{p}(\bar{z})$ and the relation between $\alpha(z,\bar{z})$ and these functions. They are given by

$$p(z) = -\frac{\kappa^2}{4z^2}\,, \qquad \bar{p}(\bar{z}) = -\frac{\kappa^2}{4\bar{z}^2}\,, \tag{2.26}$$
$$e^{2\alpha(z,\bar{z})} = \sqrt{p\bar{p}}\,. \tag{2.27}$$

These properties are then reflected on the solution of the auxiliary linear problem. By making a gauge transformation

$$\psi = \mathcal{A}\tilde{\psi}\,, \qquad \mathcal{A} = \begin{pmatrix} p^{-1/4}e^{\alpha/2} & 0 \\ 0 & p^{1/4}e^{-\alpha/2} \end{pmatrix}, \tag{2.28}$$

the equations $(\partial + B_z(\xi))\psi = 0$ and $(\bar{\partial} + B_{\bar{z}}(\xi))\psi = 0$ drastically simplify and two independent solutions for $\tilde{\psi}$ can easily be obtained as

$$\tilde{\psi}_\pm = \exp\left(\pm\frac{\kappa i}{2}\left(\xi^{-1}\ln z - \xi \ln \bar{z}\right)\right)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{2.29}$$

Now as we go around the origin once, this pair of basis functions $(\tilde{\psi}_+, \tilde{\psi}_-)$ gets transformed as[2]

$$\begin{pmatrix} \tilde{\psi}'_+ \\ \tilde{\psi}'_- \end{pmatrix} = M\begin{pmatrix} \tilde{\psi}_+ \\ \tilde{\psi}_- \end{pmatrix}, \quad M = \begin{pmatrix} e^{i\hat{p}(\xi)} & 0 \\ 0 & e^{-i\hat{p}(\xi)} \end{pmatrix}, \quad \hat{p}(\xi) = i\kappa\pi\left(\frac{1}{\xi} + \xi\right). \tag{2.30}$$

The matrix M, called the local monodromy, is an important characteristic of the LSGKP solution viewed as the local solution in the vicinity of the appropriate vertex operator.

As was already reviewed in the introduction, what was computed in our previous work is the part of the action integral called the regularized area A_{reg}. It was defined in the following way. First we split the total area A into the finite part and the divergent part as

$$A = 2\int d^2z\partial\vec{X}\cdot\bar{\partial}\vec{X} = 4\int d^2z e^{2\alpha} = A_{fin} + A_{div}\,, \tag{2.31}$$
$$A_{fin} = 4\int d^2z\,\left(e^{2\alpha} - \sqrt{p\bar{p}}\right), \qquad A_{div} = 4\int d^2z\,\sqrt{p\bar{p}}\,. \tag{2.32}$$

Then, by using the equation of motion (2.14) and evaluating a certain boundary integral, A_{fin} can be rewritten as

$$A_{fin} = 2A_{reg} + \pi(N-2)\,, \tag{2.33}$$
$$A_{reg} \equiv \int d^2z\,\left(e^{2\alpha} + p\bar{p}\,e^{-2\alpha} - 2\sqrt{p\bar{p}}\right). \tag{2.34}$$

[2]In our previous work [51], the quantity $\hat{p}(\xi)$ was denoted by $\rho(\xi)$.

For the three point function of our interest, N should be set to 3.

As explained carefully in our previous paper, since LSGKP string is completely folded, one can consider the worldsheet of half of the folded string which can be taken as the upper half plane and then smoothly extend it to the whole complex plane to account for the contribution of the other half. The net result is that, as far as the computation of the area is concerned, we may forget about the effect of folding.

The first step in computing A_{reg} is to reexpress this area integral in terms of certain contour integrals by developing a generalization of the Riemann bilinear identity. Let us briefly recall the basic idea. By introducing a convenient variable

$$\hat{\alpha} \equiv \alpha - \frac{1}{4} \ln p\bar{p} \,, \tag{2.35}$$

which vanishes at each vertex insertion point z_i, the regularized area can be rewritten as

$$A_{reg} = \frac{i}{2} \int \lambda dz \wedge \omega \tag{2.36}$$

$$\lambda \equiv \sqrt{p(z)} \,, \qquad \omega \equiv u d\bar{z} + v dz \tag{2.37}$$

$$u = 2\sqrt{\bar{p}}(\cosh 2\hat{\alpha} - 1) \,, \qquad v = \frac{1}{\sqrt{p}} (\partial \hat{\alpha})^2 \,. \tag{2.38}$$

We have added the vdz part (which does not contribute to the area) with the property $\partial u = \bar{\partial} v$ to make ω a closed 1-form. Then introducing the integral of $\lambda(z)$ as

$$\Lambda(z) = \int_{z_0}^{z} \lambda(z') dz' \,, \tag{2.39}$$

the area integral can be converted, by the Stokes theorem, into a contour integral along a boundary ∂D of a certain region D as

$$A_{reg} = \frac{i}{4} \int_D d\Lambda \wedge \omega = \frac{i}{4} \int_D d(\Lambda \omega) = \frac{i}{4} \int_{\partial D} \Lambda \omega \,. \tag{2.40}$$

To specify D, one needs to know the analytic structure of the function $\Lambda(z)$, which in turn is dictated by that of $p(z)$. As was shown in (2.26), $p(z)$ should behave around each vertex insertion point z_i as

$$p(z) \overset{z \to z_i}{\sim} \frac{\delta_i^2}{(z - z_i)^2} \,, \qquad (i = 1, 2, 3) \tag{2.41}$$

for a LSGKP string with $\delta_i^2 = -\kappa_i^2/4$, and in the case of three point function it is given explicitly by

$$p(z) = \left(\frac{\delta_1^2 z_{12} z_{13}}{z - z_1} + \frac{\delta_2^2 z_{21} z_{23}}{z - z_2} + \frac{\delta_3^2 z_{31} z_{32}}{z - z_3} \right) \frac{1}{(z - z_1)(z - z_2)(z - z_3)} \,, \tag{2.42}$$

$$z_{ij} \equiv z_i - z_j \,.$$

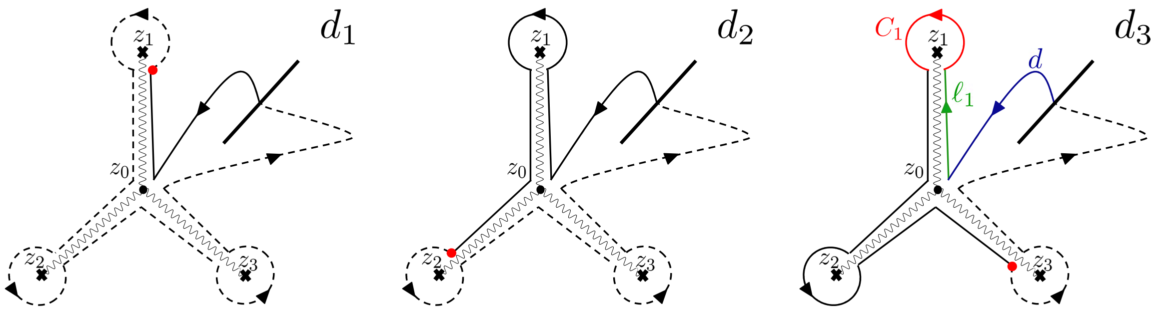

Figure 2.1: Definitions of the contours d_1, d_2 and d_3. The solid lines are on the first sheet, while the dotted lines are on the second sheet. In d_3 we indicate its "components" d, C_1 and ℓ_1. Other C_i's and ℓ_i's are defined similarly in the vicinity of z_i.

From this one can deduce that $\Lambda(z)$ has three logarithmic branch cuts running from the singularities at z_i and one square root cut connecting two zeros of $p(z)$. Thus, it is convenient to take D as the double cover of the worldsheet with appropriate boundary ∂D , on which $\Lambda(z)$ is single-valued. To evaluate the contour integral (2.40), we developed an extension of the Riemann bilinear identity, applicable in the presence of logarithmic cuts as well as the usual square root cuts, which expresses (2.40) in terms of products of simple contour integrals along basic closed paths. We will not display the general form of this identity here[3] but simply recall the final expression of A_{reg} after the application of this identity. In the case of LSGKP strings, it gets simplified to

$$A_{reg} = \frac{\pi}{12} + \frac{i}{4} \sum_{j=1}^{3} \int_{C_j} \lambda dz \int_{d_j} (u d\bar{z} + v dz) \,. \tag{2.43}$$

Here C_j is a small circle around the singularity at z_j and the integrals $\int_{C_j} \lambda dz$ can be easily evaluated. On the other hand, d_j is a contour which starts from $\widehat{z}_j$, which is a point on the second sheet right below z_j, goes around the logarithmic cuts counterclockwise and reaches z_j on the first sheet, as shown in figure 2.1. Evaluation of the integrals $\int_{d_j} (u d\bar{z} + v dz)$ constitutes the major non-trivial task.

An important observation is that such an integral along d_j is composed of more elementary units, namely integrals along paths each of which connects a pair of singular points $\{z_i, z_j\}$. Thus, one may expect that the wanted information can be extracted from the properties of the eigenfunctions of the auxiliary linear problem around the singularities z_i and along such paths.

As already shown in (2.30) for the LSGKP solution itself, the behavior around z_i is characterized by the local monodromy matrix M_i with a unit determinant. Each M_i can

[3]Later in section 5.1, where we evaluate the divergent part of the area, we will need to recall the full form of the identity.

be separately diagonalized as

$$M_i = \begin{pmatrix} e^{i\hat{p}_i(\xi)} & 0 \\ 0 & e^{-i\hat{p}_i(\xi)} \end{pmatrix} , \tag{2.44}$$

where $\hat{p}_i(\xi)$ is of the form given in (2.30). The local solutions of the auxiliary linear problem which form the eigenvectors of M_i take the form

$$i_\pm \sim \exp\left[\pm\left(\frac{1}{\xi}\int\sqrt{p(z)}dz + \xi\int\sqrt{\bar{p}(\bar{z})}d\bar{z}\right)\right] , \tag{2.45}$$

which can be normalized as $\langle i_+, i_-\rangle = 1$. In general M_i's are not simultaneously diagonalizable, due to the (unknown) non-trivial global behavior of the solutions of the auxiliary linear problem. Nevertheless, the simple global relation $M_1M_2M_3 = 1$, expressing the triviality of the monodromy around the entire worldsheet, is enough to restrict the form of M_i sufficiently (although not completely). This then allows one to compute the eigenvectors $i_\pm$ and further their $SL(2)$-invariant products $\langle i_+, j_+\rangle, \langle i_-, j_-\rangle, \langle i_+, j_-\rangle$ in terms of $\hat{p}_i(\xi)$'s. Although these expressions still contain a few undetermined coefficients, they cancel in certain products of $\langle i, j\rangle$ and we obtain definite expressions. One such example is $\langle 2_-, 1_+\rangle\langle 1_-, 2_+\rangle$, or taking the logarithm for convenience,

$$\log\langle 2_-, 1_+\rangle + \log\langle 1_-, 2_+\rangle = \log\left(\frac{\sin\frac{\hat{p}_1-\hat{p}_2+\hat{p}_3}{2}\sin\frac{-\hat{p}_1+\hat{p}_2+\hat{p}_3}{2}}{\sin\hat{p}_1\sin\hat{p}_2}\right) . \tag{2.46}$$

Later we will describe how to obtain the information of each individual invariant from the combination of this form.

Having learned that we can express (certain combinations of) the products $\langle i, j\rangle$ in terms of $\hat{p}_i(\xi)$ characterizing local LSGKP behavior, we now wish to compute $\langle i, j\rangle$ from a different more dynamical point of view, namely from the actual (global) solutions of the auxiliary linear problem. Clearly, in the absence of the saddle point solution for the three point function, this cannot be done directly. However, it turned out to be sufficient to formally solve the auxiliary linear problem in powers of ξ and $1/\xi$, that is by using the so-called "WKB expansions". In applying this method, the meaningful classification of the local behavior of the solutions is not by the overall signs of the exponent of the eigenfunctions, as in $i_\pm$ of (2.45), but by the actual signs of the real part of the exponent. In other words, what is important is whether the solution increases or decreases exponentially as it approaches z_i. In particular, the exponentially decreasing "small" solution, denoted by s_i, is important as it is not contaminated by the "big" component and is hence unambiguous.

Skipping all the somewhat tedious details, the end result of the WKB analysis around $\xi = 0$ is the formula for the $SL(2)$-invariant product of small solutions[4] $\langle s_i, s_j \rangle$. To illustrate it with a concrete example, consider the case where the branch structure of $\sqrt{p(z)}$ at z_i is such that on the first sheet

$$\sqrt{p(z)} \overset{z \sim z_i}{\sim} \frac{i\hat{\kappa}_i}{z - z_i}\,, \qquad \hat{\kappa}_i = \left\{ \begin{array}{ll} \kappa_i & \text{for } i = 1,3 \\ -\kappa_i & \text{for } i = 2 \end{array} \right. . \tag{2.47}$$

Then, if $\operatorname{Im}\xi > 0$, the small solutions at z_i can be identified with the $i_\pm$ solutions as $s_1 \sim 1_+, s_2 \sim 2_-, s_3 \sim 3_+$ and the product $\langle s_i, s_j \rangle$ is given by the formula like

$$\langle s_1\,, s_2 \rangle = \exp\left[\left(\xi^{-1} \int_{z_1}^{z_2} \sqrt{p} dz + \xi \int_{z_1}^{z_2} \sqrt{\bar{p}} d\bar{z} \right) + \frac{\xi}{2} \left(\int_{z_1}^{z_2} u d\bar{z} + v dz \right) + \cdots \right] . \tag{2.48}$$

A remarkable fact is that in the exponent the type of contour integral of our interest $\int (u d\bar{z} + v dz)$ makes its appearance. Moreover, as described in our previous paper, one can make the contour of this integral to coincide with the d_j's (recall figure 2.1) by making judicious ratios of $\langle s_i, s_j \rangle$'s. For example, the one along the contour d_1 can be produced in

$$\frac{\langle s_2\,, s_3 \rangle}{\langle s_2\,, s_1 \rangle \langle s_1\,, s_3 \rangle} = \exp\left[\frac{1}{\xi} \int_{d_1} \lambda dz + \xi \int_{d_1} \sqrt{\bar{p}} d\bar{z} + \frac{\xi}{2} \left(\int_{d_1} u d\bar{z} + v dz \right) + \cdots \right] . \tag{2.49}$$

This shows that the computation of the integrals $\int_{d_j} (u d\bar{z} + v dz)$, and hence the regularized area, is now reduced to that of $\langle s_i, s_j \rangle$.

At this point, let us recall that we already have some important explicit knowledge of such $SL(2)$ invariant products, although not directly of $\langle s_i, s_j \rangle$. It is the information about certain products of $\langle i, j \rangle$'s, such as $\langle 1_+, 2_- \rangle \langle 1_-, 2_+ \rangle$ shown in (2.46). Now the problem is how to extract the information of the individual factors $\langle 1_+, 2_- \rangle$ and $\langle 1_-, 2_+ \rangle$ and relate them to $\langle s_1, s_2 \rangle$. The key to the solution of this problem is to recognize that which of the $i_\pm$ is identified with the small solution s_i depends crucially on the sign of the imaginary part[5] of ξ. As we already mentioned, if $\operatorname{Im}\xi > 0$ the identification is $1_+ \sim s_1$ and $2_- \sim s_2$. However, for $\operatorname{Im}\xi < 0$ the identification is reversed, namely $1_- \sim s_1$ and $2_+ \sim s_2$. Therefore, $\langle 1_+, 2_- \rangle$ and $\langle 1_-, 2_+ \rangle$ must be identified with $\langle s_1, s_2 \rangle$ in the domains $\operatorname{Im}\xi > 0$ and $\operatorname{Im}\xi < 0$ respectively. This analyticity property strongly characterizes the logarithms of a part of $\langle 1_+, 2_- \rangle$ and $\langle 1_-, 2_+ \rangle$ which are regular at $\xi = 0, \infty$. Applying the Wiener-Hopf decomposition

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\xi' \frac{1}{\xi' - \xi} \left(F(\xi') + G(\xi') \right) = \left\{ \begin{array}{ll} F(\xi)\,, & (\operatorname{Im}\xi > 0) \\ -G(\xi)\,, & (\operatorname{Im}\xi < 0) \end{array} \right. , \tag{2.50}$$

[4]There is a similar formula for the expansion around $\xi = \infty$.

[5] This is because the residue of the pole of $\sqrt{p(z)}$ at the singularity is pure imaginary and hence the change of the sign of $\operatorname{Im}\xi$ swaps the small and the big solutions.

with[6] $F = \log\langle 2_-, 1_+\rangle_{n.s.}$ and $G = \log\langle 1_-, 2_+\rangle_{n.s.}$, we can extract F and G separately from the information of the sum $F + G$ given in (2.46). Using similar procedures we can compute the regular part of all the invariant products $\langle s_i, s_j\rangle$ and the combinations such as (2.49). Finally, extracting the part proportional to ξ we can express the integrals $\int_{d_j}(ud\bar{z} + vdz)$ solely in terms of the parameters κ_i of the LSGKP external strings. The final formula for the regularized area, which is valid for all the regimes of the parameters κ_i, is given by

$$
\begin{aligned}
A_{reg} = \frac{\pi}{12} + \pi\Big[& -\kappa_1 K(\kappa_1) - \kappa_2 K(\kappa_2) - \kappa_3 K(\kappa_3) \\
& + \frac{\kappa_1+\kappa_2+\kappa_3}{2} K(\frac{\kappa_1+\kappa_2+\kappa_3}{2}) \\
& + \frac{|-\kappa_1+\kappa_2+\kappa_3|}{2} K(\frac{|-\kappa_1+\kappa_2+\kappa_3|}{2}) \\
& + \frac{|\kappa_1-\kappa_2+\kappa_3|}{2} K(\frac{|\kappa_1-\kappa_2+\kappa_3|}{2}) \\
& + \frac{|\kappa_1+\kappa_2-\kappa_3|}{2} K(\frac{|\kappa_1+\kappa_2-\kappa_3|}{2})\Big] ,
\end{aligned} \tag{2.51}
$$

where the function $K(x)$ is defined as

$$
K(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\theta\, e^{-\theta} \log\left(1 - e^{-4\pi x\cosh\theta}\right) . \tag{2.52}
$$

3 Method for evaluating the contribution of vertex operators

The major problem left unsolved in our previous work is the evaluation of the contributions of the vertex operators which generate the LSGKP strings from the boundary of AdS spacetime. In this section, we first give a general discussion of why it is a difficult problem and give an outline of how we are going to solve it. Subsequently, we will implement the idea more concretely and develop a method which can be used for general semi-classical correlation functions. We will first apply it to two-point functions in section 4 and check that it produces expected results. Then finally in section 5 three point functions will be treated by the same method and we will obtain natural complete results.

3.1 General strategies

There are two obvious obstacles in the evaluation of the vertex operator contributions. First, as discussed in detail in our previous paper, the precise form of the conformally

[6]Subscript "n.s." denotes the part regular (nonsingular) at $\xi = 0, \infty$.

invariant vertex operator corresponding to the GKP string is not known. The curvature of the AdS background, the effect of which is certainly felt by the macroscopic strings of our interest, inherently couples the left and the right moving modes on the worldsheet and hence the two Virasoro symmetries are not directly associated with the holomorphic and the anti-holomorphic dependence of the fields. Furthermore, even on the boundary of the AdS space, the GKP string explicitly depends on the σ coordinate and consists of infinite number of Fourier modes. This indicates that the vertex operator which creates such a string state is expected to contain infinite number of σ derivatives. Thus, although it is easy to find operators with the correct global quantum numbers such as the dilatation charge and the spin, it is quite difficult to construct the vertex operator with the right conformal property describing such an extended string state. The second difficulty is that we do not know any of the classical solutions which serve as the saddle configurations for the three and higher point correlation functions. Thus, even if we succeed in constructing correct vertex operators, we do not have the solutions with which to evaluate their values.

How can one overcome these apparently serious difficulties? Below we give an outline of our strategy to solve these problems by making use of the so-called finite gap method of integrable theories.

As we have already argued in our previous work, by using the state-operator correspondence, we can evaluate the effect of the vertex operators through the semi-classical wave functions, which can be constructed in principle by solving the Hamilton-Jacobi equation. When the saddle point approximation is adequate, the state-operator correspondence can be expressed as

$$V[q_*(z=0)]e^{-S_{q_*}(\tau<\tau_0)} = \Psi[q_*(\tau_0)]\,, \tag{3.1}$$

where q_* denotes the saddle point configuration, $V[q_*(z=0)]$ is the value of the vertex operator inserted at the origin of the worldsheet $z = e^{\tau+i\sigma} = 0$, corresponding to the local cylinder time $\tau = -\infty$, the factor $\exp[-S_{q_*}(\tau < \tau_0)]$ is the amplitude to develop into the state on the small circle with radius e^{τ_0} and $\Psi[q_*(\tau_0)]$ is the semi-classical wave function describing the state on that circle[7]. The variable q can be the string coordinates $X_\mu(\sigma)$ or any other variables which can specify the configuration. In particular, if we can construct the action-angle variables (J_n, θ_n) of the system and use $\{\theta_n\}$ as q, then the (Euclidean)

[7] At the end of this subsection, we will discuss this formula also from the point of view of the renormalization of the vertex operator.

action and the wave function can be expressed simply as

$$S_\theta = -\int d\tau \left(i \sum_n J_n \partial_\tau \theta_n - \mathcal{E}(\{J_n\}) \right) , \tag{3.2}$$

$$\Psi[\theta] = \exp\left(i \sum_n J_n \theta_n - \mathcal{E}(\{J_n\})\tau \right) , \tag{3.3}$$

where both $J_n \partial_\tau \theta_n$ and $\mathcal{E}(\{J_n\})$, the worldsheet energy, are constant. Our main idea is to construct and evaluate the wave functions in the representation (3.3) above.

Before explaining what we need to do to achieve this, let us make an important remark concerning the evaluation of the action. As it will be shown in section 4, if we can evaluate the action also in terms of the action-angle variables, as in (3.2), the evaluation of the two point function becomes systematic and simple. In particular the cancellation of the divergences between the wave functions and the action becomes automatic. However, for the calculation of the action for the three point function of the LSGKP strings performed in our previous paper [51] and in section 5.1, the action is expressed and computed in terms of the string coordinates $X_\mu(\sigma)$, not the angle variables. In such a case, in order to combine properly with the contribution of the wave functions evaluated in terms of the angle variables, one must make a canonical transformation and, as discussed in Appendix D, this in general produces an additional contribution. Fortunately, in the case of the GKP string, we show in Appendix E that such an extra contribution does not affect the final result.

Now let us describe the three problems to be solved in order to actually construct and evaluate the wave functions in terms of the action-angle variables.

(1) First problem is the construction of the action-angle variables for the string in Euclidean AdS_3. It turns out that the direct construction through the Hamilton-Jacobi method is extremely difficult, if not impossible[8]. However, in the case of the so-called finite gap solutions, to which the GKP string belongs, we can make use of the Sklyanin's "separation of variable" method [58] to construct the action-angle variables from the positions of the poles of the suitably normalized eigenvector (called the Baker-Akhiezer vector) of the auxiliary linear problem. In fact, we will argue that in order to keep full grasp of all the dynamical degrees of freedom one should consider the problem from the perspective of the "infinite gap solutions" and understand the finite gap solutions as suitable limits. This will be discussed in detail in sections 3.2.2 and 3.2.3.

[8]However, for a particle in Euclidean AdS_3, such a construction is possible and is instructive. See Appendix G for some detail.

(2) The Sklyanin's method referred to above identifies the action-angle variables formally for finite gap solutions. What we need to know are the *values* of the action-angle variables for the specific GKP string solutions. For this purpose, we first need to re-analyze the GKP solution and its large spin limit using the finite gap method, rather than in the framework of Pohlmeyer reduction we have been employing. In contrast to the well-studied cases of strings in $AdS_3 \times S_1$ (or in $R \times S^3$), the finite gap method for the case of the string in (Euclidean) AdS_3 has not been fully developed. This is largely because the structure of the associated spectral curve is more involved due to the different form of the Virasoro conditions imposed on the AdS_3 part. This point is clarified in section 3.2.1. With this structure duly understood, we then develop a powerful method of computing the values of the angle variables based on the use of the global symmetry transformations. The basic idea is to compute the angle variables of a solution $\mathbb{X}$ of our interest *relatively* to those of a suitable fixed reference solution $\mathbb{X}^{\rm ref}$. Since $\mathbb{X}$ can be obtained from $\mathbb{X}^{\rm ref}$ by a global $SL(2)_L \times SL(2)_R$ symmetry transformation as $\mathbb{X} = V_L \mathbb{X}^{\rm ref} V_R$, all we have to know is how the angle variables shift under such a transformation. As will be shown in section 3.3.1, this can be computed explicitly in terms of the parameters of the global transformation and the components of the normalization vector $\vec{n}$, which determines the locations of the poles of the Baker-Akhiezer vector.

(3) The last problem to solve is how to construct the suitable quantum wave functions, corresponding to the vertex operators, using the classical data on the action-angle variables obtained by the method described above. Here again the global symmetry transformations play the key role.

Let us explain this explicitly for the case of the two point function of the form $\langle V(0)V(x)\rangle$, where the vertex operator $V(0)$ corresponds to a conformal primary operator on the SYM side inserted at the origin of the boundary of AdS_3 and $V(x)$ is the same operator inserted at x on the boundary. In the saddle point approximation, this is evaluated as $V(0)\big|_{\mathbb{X}} V(x)\big|_{\mathbb{X}}$. Before proceeding, let us clarify here how the vertex insertion points should be identified with the behavior of the saddle point solution on the boundary. For the LSGKP solution we are considering, the string configuration is extended, not point-like, even on the boundary and this appears to make such an identification ambiguous. However, there is a natural and unique choice. As we already mentioned under item (2), a general solution can be obtained from the reference solution $\mathbb{X}^{\rm ref}$ (2.9) by a global symmetry transformation. Clearly the spin of the reference solution is defined with respect to the rotation in the x_1-x_2 plane around the origin $(0, 0)$, which is the $\sigma = 0$ point on the string. Under the global symmetry transformation, such a point $X_\mu(\sigma = 0)$ transforms covariantly and hence the two point function behaves correctly under the con-

formal transformation of the boundary coordinates. Therefore the vertex insertion point should be identified with the point at $\sigma = 0$.

Now let us first evaluate $V(0)\big|_{\mathbb{X}}$ using the corresponding wave function constructed in terms of the angle variables. Recall that such angle variables depend on the choice of the normalization vector $\vec{n}$ of the Baker-Akhiezer function. As it will be shown in section 3.3.2, this normalization vector is uniquely determined by the requirement that the wave function is unchanged under the special conformal transformation, namely that it is a conformal primary. Once $\vec{n}$ is fixed, then we can compute the (shift of the) angle variables by the method described in the item (2) above.

Next consider the evaluation of $V(x)|_{\mathbb{X}}$. The vertex operator $V(x)$ is of course different from $V(0)$. In particular, $V(x)$ is not invariant under the special conformal transformation around the origin which leaves $V(0)$ invariant. Thus it appears that we have to re-analyze the condition for the normalization vector $\vec{n}$ and the proper form of the corresponding wave function, which is quite cumbersome. We can actually circumvent this procedure by noting that $V(x)$ is obtained from $V(0)$ by a translation, which is again a global symmetry transformation. Symbolically, $V(x) = T_x V(0)$. Further, we can evaluate this on the solution $\mathbb{X}$ in the manner $V(x)\big|_{\mathbb{X}} = (T_x V(0))\big|_{\mathbb{X}} = V(0)\big|_{T_x^{-1}\mathbb{X}}$. In the last step, instead of evaluating the shifted vertex operator on the solution $\mathbb{X}$, equivalently we evaluate the unshifted operator at the origin on the inversely translated solution $T_x^{-1}\mathbb{X}$, which is quite easy to obtain. This method has another important advantage that we can compute the shift of the angle variables relative to the same reference solution $\mathbb{X}^{\mathrm{ref}}$ which emanates from the origin of the boundary. See figure 4.1. The details of this method will be explained in section 3.3.

So in this way, we can compute the contribution of the vertex operator for the LSGKP string inserted at any point from the local behavior of the solution at that point. It should be clear that the method is quite general and can be applied to correlation functions for any string states describable by the finite gap method.

Before ending this subsection, we wish to discuss the structure of the divergences that appear in the evaluation and how they cancel in the end result. The divergences that we encounter for the correlation functions are all of ultraviolet origin and hence quite local. Therefore, their properties should be the same as in the string theory in flat spacetime, and we need to use "normal-ordered" (or "renormalized") vertex operators to get finite results. This is all standard but in the saddle point computation the structure is somewhat unfamiliar. The first source of divergences is the action evaluated on the saddle point configuration. There are charges at the vertex insertion points and the action

contains the self-energy of each local charge, which is log-divergent in two dimensions. The second source of divergences is the contribution from the naive (unrenormalized) vertex operators, such as $V = e^{ikX}$ for a tachyon in flat space, evaluated on the saddle. As one can easily check in the case of free string, this contribution *over-compensates* the one from the action and the net result is still divergent in the exponent. Finally to cancel this remainder we need to introduce divergent renormalization factor for each vertex. In other words, if we prepare the renormalized vertex operators $: V_i :$ from the out set, their log's $\log : V_i[X_*] :$ on the saddle configuration X_* are divergent in such a way that the correlation function $e^{-S[X_*]}(: V_1[X_*] :: V_2[X_*] : \cdots : V_n[X_*] :)$ becomes finite.

Now from this point of view, the saddle point version of the state-operator correspondence (3.1) states that the wave function $\Psi[q_*(\tau_0)]$ gives nothing but the contribution of the *renormalized* vertex operator evaluated on the saddle. Hence $\log \Psi$ should contain divergences which precisely kill those from the action. We shall demonstrate this explicitly when we compute the final three point function in section 5.

3.2 Construction of action-angle variables for a string in Euclidean AdS_3

As explained in the previous subsection, instead of trying to evaluate the contribution of the vertex operators directly, we will deal with the wave functions which are related to the vertex operators through the state-operator correspondence. The wave functions in the classical limit are given by the solutions of the Hamilton-Jacobi (H-J) equation. Although the solutions can be obtained rather easily for simple systems, such as a string moving in flat space, it is quite difficult to solve the H-J equation for non-linear systems, such as a string in Euclidean AdS_3. However, as we shall see, the integrability of the system will allow us to construct the action-angle variables, in terms of which the H-J equation simplifies enormously. In this subsection we will first discuss the classical integrability of a string in Euclidean AdS_3 in the framework of the spectral curve method, clarifying the differences from the case of a string in $AdS_3 \times S^1$, which was studied in [21]. Then a powerful method of constructing the action-angle variables for the finite gap solutions will be explained and describe how they will be of use for the computation of correlation functions.

3.2.1 Analytic structure of the quasi-momentum and the spectral curve

The classical motion of a string in $AdS_5 \times S^5$ is well-known to be integrable. Moreover, the integrability persists even when the motion of a string is restricted to certain subspaces of $AdS_5 \times S^5$. The Euclidean AdS_3, the space of our interest, is among such class of subspaces. Although the integrability of a string on a closely related space, namely $AdS_3 \times S^1$, has been well-studied in [21], the structure for case of the Euclidean AdS_3 exhibits several important differences, which we shall clarify in the analysis below.

The classical integrability of the string in AdS_3 is most apparent when we reformulate the equation of motion as the zero-curvature condition for a one-parameter family of connections viz. Lax connections. It is given by

$$\left[\partial + J_z^r(x)\,, \bar{\partial} + J_{\bar{z}}^r(x)\right] = 0\,, \tag{3.4}$$

$$J_z^r(x) \equiv \frac{1}{1-x} j_z\,, \qquad J_{\bar{z}}^r(x) \equiv \frac{1}{1+x} j_{\bar{z}}\,, \tag{3.5}$$

$$j_z = \mathbb{X}^{-1}\partial\mathbb{X}\,, \qquad j_{\bar{z}} = \mathbb{X}^{-1}\bar{\partial}\mathbb{X}\,, \tag{3.6}$$

$$\mathbb{X} = \begin{pmatrix} X_+ & X \\ \bar{X} & X_- \end{pmatrix}\,. \tag{3.7}$$

Here j_z and $j_{\bar{z}}$ are the components of the left-invariant current j, which transforms only under $SL(2)_R$. To emphasize this transformation property, we denote by J_z^r and $J_{\bar{z}}^r$, with the subscript r, the components of the flat Lax connection made from the "right current" j. The complex variable x denotes the familiar spectral parameter. Similarly, one can construct another one-parameter family of flat connections J^l from the "left current" $l = d\mathbb{X}\mathbb{X}^{-1}$ in the following way:

$$\left[\partial + J_z^l(x)\,, \bar{\partial} + J_{\bar{z}}^l(x)\right] = 0\,, \tag{3.8}$$

$$J_z^l(x) \equiv \frac{x}{1-x} l_z\,, \qquad J_{\bar{z}}^l(x) \equiv -\frac{x}{1+x} l_{\bar{z}}\,, \tag{3.9}$$

$$l_z = \partial\mathbb{X}\mathbb{X}^{-1}\,, \qquad l_{\bar{z}} = \bar{\partial}\mathbb{X}\mathbb{X}^{-1}\,. \tag{3.10}$$

These two connections are related to each other by the gauge transformation of the form $\mathbb{X}(d + J^r)\mathbb{X}^{-1} = d + J^l$. Furthermore, as shown in Appendix A of [54], they are gauge equivalent also to the connections constructed from the Pohlmeyer reduction (2.16) under the identification of the spectral parameters $x = (1-\xi^2)/(1+\xi^2)$. Although the Pohlmeyer reduction is suitable for the evaluation of the area, use of the connections $J^{r,l}$ will be more advantageous for the construction of the action-angle variables.

One of the manifestations of integrability is the existence of an infinite number of conserved charges. They are constructed from the path-ordered exponential of the connection $J^r(x)$ along a closed cycle around an insertion point of a vertex operator, called

the monodromy matrix:

$$\Omega(x; z_0) = \mathcal{P} \exp \left(- \oint J_z^r(x) dz + J_{\bar{z}}^r(x) d\bar{z} \right) . \tag{3.11}$$

As indicated, the matrix Ω depends on the base point of the closed cycle z_0. By virtue of the flatness of the connection, an expansion of $\Omega(x)$ as a function of x around some point yields an infinite number of conserved charges as coefficients. In particular, expansions around $x = \infty$ and $x = 0$ yield global charges, corresponding to $SL(2)_R$ and $SL(2)_L$ respectively, at the leading order in the following way:

$$\Omega(x; z_0) = \mathbf{1} - \frac{2\pi i}{\sqrt{\lambda} x} Q_R + O(x^{-2}) \qquad (x \to \infty) , \tag{3.12}$$

$$\mathbb{X}(z_0) \Omega(x; z_0) \mathbb{X}^{-1}(z_0) = \mathbf{1} + \frac{2\pi i x}{\sqrt{\lambda}} Q_L + O(x^2) \qquad (x \to 0) , \tag{3.13}$$

where the matrices Q_R and Q_L are given by

$$Q_R \equiv \frac{i\sqrt{\lambda}}{2\pi} \oint (j_z dz - j_{\bar{z}} d\bar{z}) , \qquad Q_L \equiv \frac{i\sqrt{\lambda}}{2\pi} \oint (l_z dz - l_{\bar{z}} d\bar{z}) . \tag{3.14}$$

Quantities independent of the base point z_0 can be extracted from the eigenvalues of Ω. Since $\det \Omega = 1$, these eigenvalues are of the form

$$u(x; z_0) \Omega(x; z_0) u(x; z_0)^{-1} \sim \begin{pmatrix} e^{i\hat{p}(x)} & 0 \\ 0 & e^{-i\hat{p}(x)} \end{pmatrix} , \tag{3.15}$$

where $u(x; z_0)$ is the matrix which diagonalizes Ω and $\hat{p}(x)$ is the quasi-momentum[9]. One can choose the branch of the logarithm appropriately so that $\hat{p}(x)$ exhibits the following asymptotic behavior, reflecting the asymptotics of Ω around $x = \infty$ and $x = 0$:

$$\hat{p}(x) = -\frac{2\pi}{\sqrt{\lambda} x} R + O(x^{-2}) \qquad (x \to \infty) , \tag{3.16}$$

$$\hat{p}(x) = 2\pi m + \frac{2\pi x}{\sqrt{\lambda}} L + O(x^2) \qquad (x \to 0) . \tag{3.17}$$

Here the conserved charges R and L are the (upper) eigenvalues of Q_R and Q_L respectively and m is an integer. Although m is nonvanishing for a general string state, we will focus on the class of solutions with $m = 0$, to which the GKP string belongs[10].

To discuss other conserved charges, it is important to study the analytic properties of $\hat{p}(x)$. Such analytic structures are encoded in the spectral curve defined by

$$\Gamma \; : \quad \Gamma(x, y) \equiv \det \left(y\mathbf{1} - \Omega(x; z_0) \right) = 0 , \tag{3.18}$$

[9]The quasi-momentum is customarily denoted by $p(x)$ without a hat [21]. However, to distinguish it from the function $p(z)$ which appears in the Pohlmeyer reduction, we will denote it by $\hat{p}(x)$.

[10]Generalization to solutions with $m \neq 0$ is straightforward.

Figure 3.1: Analytic structure of the spectral curve of a string on $AdS_3 \times S^1$. The wavy lines denote square root cuts. There are essential singularities at $x = \pm 1$, corresponding to simple poles in $\hat{p}(x)$. The node-like points, denoted by crosses, accumulate to these points.

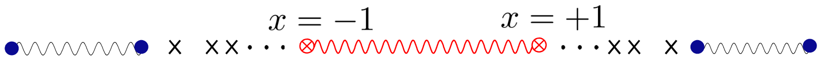

Figure 3.2: Analytic structure of the spectral curve of a string on Euclidean AdS_3. Compared to the case of $AdS_3 \times S^1$, the singularities at $x = \pm 1$ are weakened to "half-pole" type and a singular square root cut runs between these two points.

which is equivalent to $\left(y - e^{i\hat{p}(x)}\right)\left(y - e^{-i\hat{p}(x)}\right) = 0$. As we shall show, the spectral curve Γ has three kinds of analytic structures, namely essential singularities, cusp-like points and node-like points (see figures 3.1 and 3.2).

Let us first focus on the essential singularities. As we shall show shortly, for a string on Euclidean AdS_3 the structure of essential singularities turned out to be somewhat complicated. Therefore it is instructive to consider first a simpler case of a string in $AdS_3 \times S^1$, where the motion in S^1 provides a constant contribution to the Virasoro conditions, as we discuss below. For this case, it is known that the essential singularities arise at $x = \pm 1$, where the Lax connection (3.5) becomes singular [21]. To see this, recall the definition of the monodromy matrix (3.11). Near $x = \pm 1$, it behaves as

$$\Omega(x; z_0) = \mathcal{P} \exp\left[-\oint dz \frac{j_z}{1-x} + O((x-1)^0)\right] \qquad (x \to 1)\,, \tag{3.19}$$

$$\Omega(x; z_0) = \mathcal{P} \exp\left[-\oint dz \frac{j_{\bar{z}}}{1+x} + O((x+1)^0)\right] \qquad (x \to -1)\,. \tag{3.20}$$

The Virasoro conditions for the entire string reads

$$\partial X^\mu \partial X_\mu = \frac{1}{2} \mathrm{Tr}\,(j_z j_z) = \kappa^2\,, \qquad \bar{\partial} X^\mu \bar{\partial} X_\mu = \frac{1}{2} \mathrm{Tr}\,(j_{\bar{z}} j_{\bar{z}}) = \kappa^2\,. \tag{3.21}$$

where κ^2 denotes the contribution from the S^1 part. Diagonalization of (3.19) and (3.20) leads to

$$u(x; z_0)\Omega(x; z_0)u(x; z_0)^{-1} = \exp\left[\frac{2i\pi\kappa}{x \mp 1}\sigma_3 + O((x \mp 1)^0)\right] \qquad (x \to \pm 1)\,. \tag{3.22}$$

(3.22) clearly shows the existence of essential singularities at $x = \pm 1$ (see figure 3.1).

They correspond to the simple pole singularities of $\hat{p}(x)$ of the form

$$\hat{p}(x) = \frac{2\pi\kappa}{x \mp 1} + O(1) \qquad (x \to \pm 1)\,. \tag{3.23}$$

Let us now return to the case of the Euclidean AdS_3. In this case, in contrast to (3.21), the Virasoro conditions read

$$\partial X^\mu \partial X_\mu = \frac{1}{2}\mathrm{Tr}\,(j_z j_z) = 0\,, \qquad \bar{\partial} X^\mu \bar{\partial} X_\mu = \frac{1}{2}\mathrm{Tr}\,(j_{\bar{z}} j_{\bar{z}}) = 0\,. \tag{3.24}$$

As this corresponds to the limit of vanishing κ, we need to perform a more involved analysis to extract the behavior of $\hat{p}(x)$ around $x = \pm 1$. The result of such an analysis, carried out in Appendix A, gives the leading singular behavior of $\hat{p}(x)$ to be

$$\hat{p}(x) = \pm \frac{\kappa_\pm}{\sqrt{1 \mp x}} + O((x \mp 1)) \qquad (x \to \pm 1)\,, \tag{3.25}$$

where the constants $\kappa_\pm$ are given by

$$\kappa_+ = \frac{1}{2\pi i}\oint dz \left(\frac{1}{2}\mathrm{Tr}\,(\partial j_z \partial j_z)\right)^{1/4}, \qquad \kappa_- = \frac{1}{2\pi i}\oint d\bar{z} \left(\frac{1}{2}\mathrm{Tr}\,(\bar{\partial} j_{\bar{z}} \bar{\partial} j_{\bar{z}})\right)^{1/4}. \tag{3.26}$$

This shows that, compared to the case of $AdS_3 \times S^1$, the singularity is no longer of the simple pole type but rather a weaker "half-pole" type, with an associated branch cut between $x = +1$ and $x = -1$ as depicted in figure 3.2. For later convenience, we refer to this branch cut as the *singular cut*.

Next, let us discuss the remaining analytic structures, *i.e.* the cusp-like points and the node-like points. Both of them are defined as the zeros of the discriminant Δ_Γ of the spectral curve given by

$$\Delta_\Gamma \equiv \left(e^{i\hat{p}(x)} - e^{-i\hat{p}(x)}\right)^2\,. \tag{3.27}$$

Note that, although they are singular points of the spectral curve, the quasi-momentum $\hat{p}(x)$ is not singular at these points. They are classified according to the order of the zero. If the order of the zero is odd, *i.e.* $\Delta_\Gamma \sim (x - x_i^{(c)})^{2r+1}$, then such a point is called cusp-like. If it is even, like $\Delta_\Gamma \sim (x - x_i^{(n)})^{2r}$, it is called node-like. Around such a zero, the quasi-momentum is approximated as

$$e^{i\hat{p}(x)} \sim \pm\left(1 + \frac{\sqrt{\Delta_\Gamma}}{2}\right) \quad \Rightarrow \quad \hat{p}(x) \sim m\pi + \frac{\sqrt{\Delta_\Gamma}}{2i} \qquad m \in \mathbb{Z}\,. \tag{3.28}$$

This shows that at the cusp-like points[11] the spectral curve develops branch cuts. In what follows, we call these branch cuts *nonsingular cuts*, to distinguish them from the singular

[11] By appropriately choosing the branch of the logarithm, the integer m in (3.28) can be set to zero at the cusp-like points. We refer the reader to [59] for details.

cut connecting the points $x = \pm 1$ defined previously. Another important property of a cusp-like point is that, as shown in Proposition 7.3 in [59], the monodromy matrix at such a point always takes the form of a Jordan block, namely

$$\Omega(x_i^{(c)}) \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} . \tag{3.29}$$

Now consider the properties of the node-like points. The formula (3.28) shows that in this case the spectral curve does not develop a branch cut and such a point is characterized simply by

$$\hat{p}(x_i^{(n)}) = m_i \pi \, . \tag{3.30}$$

As concerns the form of the monodromy matrix, there are two possibilities at a node-like point. It either takes the form of a Jordan block or is proportional to the identity matrix:

$$\Omega(x_i^{(n)}) \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \pm \mathbf{1} \, . \tag{3.31}$$

From the preceding discussion we see that, for both $AdS_3 \times S^1$ and AdS_3 cases, the quasi-momentum $\hat{p}(x)$ has singularities at $x = \pm 1$ and infinite number of node-like and/or cusp-like points accumulate at $x = \pm 1$. Consequently, the spectral curve becomes non-algebraic and is difficult to study. In such a case, one often replaces the spectral curve with the a simpler curve $\hat{\Sigma}$ [22, 59–61] defined with the use of the logarithmic derivative of Ω as

$$\hat{\Sigma} \, : \hat{\Sigma}(x, y) \equiv \det \left(y \mathbf{1} - L(x; z_0) \right) = 0 \, , \tag{3.32}$$

$$u(x; z_0) L(x; z_0) u^{-1}(x; z_0) \equiv -i \frac{\partial}{\partial x} \log \left(u(x; z_0) \Omega(x; z_0) u(x; z_0)^{-1} \right) \, . \tag{3.33}$$

Following [59], we will call it the logarithmic derivative curve or a *log-curve* for short. The log-curve preserves the branch cut structure of the spectral curve and has the nice property that the eigenvectors of L is the same as those of Ω. At the same time, the essential singularities and the node-like points present on the spectral curve are removed on the log-curve. This feature makes it much easier to handle compared to the spectral curve. In fact for the so-called finite gap solutions, for which there are only a finite number of cusp-like points, the log-curve reduces to an algebraic curve [22, 60], a subject well-studied in mathematical literature. Of course one should remember that by going to the log-curve and neglecting the node-like points, some of the important information carried by the spectral curve may be lost. Despite this possible drawback, the log-curve

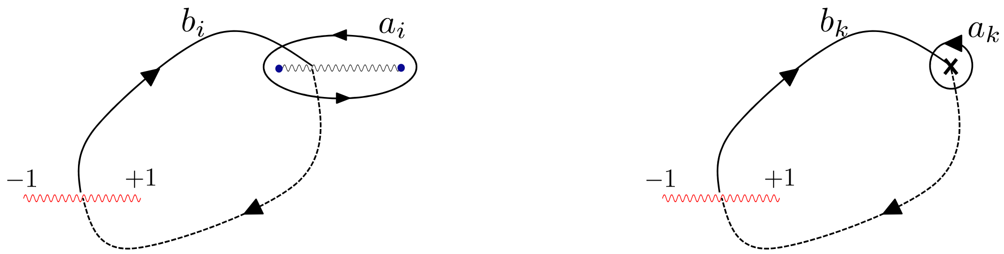

Figure 3.3: Definition of a-cycles and b-cycles. Shown on the right is the case where a non-singular cut is shrunk to a node-like point denoted by a cross.

is widely used in the literature because usually node-like points do not play important roles, as far as the construction of the finite gap solutions is concerned[12].

Now for our purpose of constructing the complete set of action-angle variables capable of describing all the states of a string in the Euclidean AdS_3, it is not appropriate to deal with the log-curve and discard the node-like points. This is because the node-like points, which can be treated as the degeneration limit of the usual branch cuts [60], correspond to the modes of the string which are not excited for a finite gap solution. In other words the angle variables for these modes do exist as dynamical variables. Therefore for us the node-like points are conceptually important and we shall study the spectral curve, not the log-curve, with the node-like points treated as a special subset of nonsingular cuts.

To extract nontrivial information from the spectral curve, we now introduce a basis of cycles on the spectral curve. As shown in figure 3.3, a convenient choice is to define the a-cycles as those which surround nonsingular cuts counterclockwise and the b-cycles as those connecting the singular cut and the nonsingular cuts. Under an appropriate choice of the branch of the logarithm and the positions of the branch cuts, the integrals of the differential $d\hat{p}$ along a- and b-cycles take the following form [60, 61]:

$$\int_{a_i} d\hat{p} = 0\,, \qquad \int_{b_i} d\hat{p} = 2\pi n_i\,, \qquad n_i \in \mathbb{Z}\,. \tag{3.34}$$

In addition to these cycles, it is convenient to introduce four more cycles a_0, a_∞, b_0 and b_∞. The cycles a_0 and a_∞ surround the points $x = 0$ and $x = \infty$ counterclockwise respectively, while b_0 and b_∞ connect the singular cut with $x = 0$ and $x = \infty$. As discussed in [59], one can treat these cycles essentially on equal footing with the other cycles. Now using the a-type cycles, one can define a set of conserved charges called *filling fractions* as

$$S_i \equiv \frac{i\sqrt{\lambda}}{8\pi^2}\int_{a_i} \hat{p}(x)dz\,, \tag{3.35}$$

[12] We will discuss this point further in 3.2.3.

where

$$z = x + \frac{1}{x} \tag{3.36}$$

is the Zhukovsky variable. As it will be discussed shortly, when interpreted appropriately as dynamical variables of a string system, $\hat{p}(x)$ and z are canonically conjugate and hence the definition (3.35) is nothing but that of an action variable. For this reason the filling fractions are of extreme importance and we shall construct the angle variables as their conjugates in section 3.2.2 below. Among the S_i's, S_0 and S_∞ are of special interest since they correspond to the global charges R and L in the following way:

$$S_0 = \frac{L}{2}\,, \qquad S_\infty = -\frac{R}{2}\,. \tag{3.37}$$

It should be remarked that the filing fractions for the node-like points vanish, since $\hat{p}(x)$ is not singular at those points:

$$S_k = \frac{i\sqrt{\lambda}}{8\pi^2}\oint_{x_k} \hat{p}(x)dz = 0\,. \tag{3.38}$$

This is consistent with the interpretation of the node-like points as representing unexcited modes of the string.

3.2.2 Action-angle variables for "infinite gap" solutions

Now we move on to the construction of the action-angle variables. For a closely related system, namely the string in $R \times S^3$, the action-angle variables have been constructed in [59, 61] by employing the so-called Sklyanin's separation of variables [58]. As we shall describe below, with some modifications this method is applicable also to the string in Euclidean AdS_3.

Before proceeding to the details, we wish to emphasize an important difference between [59, 61] and our discussion below. While the works [59, 61] focused only on the finite gap solutions, namely the solutions with a finite number of cusp-like points, we shall deal with an enlarged category of solutions which have an infinite number of cusp-like points but no node-like points (see figure 3.4). We will refer to such solutions as "infinite gap" solutions. We exclude the presence of node-like points in the above definition because such a point can be universally described by shrinkage of a branch cut between two cusp-like points. The framework of infinite gap solutions is extremely important and useful in controlling the complete degrees of freedom of the string. Of course all the other solutions, including

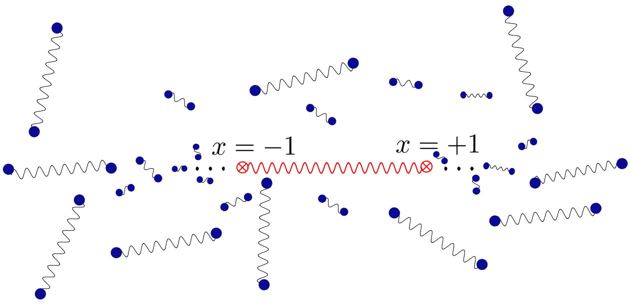

Figure 3.4: Schematic picture of the spectral curve of an infinite gap solution.

the finite gap solutions, can be obtained by certain degeneration limits[13] of such infinite gap solutions.

Now let us describe the Sklyanin's method, as applied to a string in Euclidean AdS_3. It is a powerful method for constructing canonically conjugate variables and is known to be applicable to a wide variety of integrable systems possessing Lax representation. The main object of concern is the eigenvector $\vec{\psi}$, called the Baker-Akhiezer vector, of the monodromy matrix Ω, satisfying the eigenvalue equation of the form

$$\Omega(x;\tau,\sigma)\vec{\psi}(x;\tau,\sigma) = e^{i\hat{p}(x)}\vec{\psi}(x;\tau,\sigma)\,. \tag{3.39}$$

Actually, it is of crucial importance to consider the normalized Baker-Akhiezer vector $\vec{h}(x;\tau)$, defined to be proportional to $\vec{\psi}(x;\tau,\sigma=0)$ and normalized by the condition

$$\vec{n}\cdot\vec{h} = n_1 h_1 + n_2 h_2 = 1\,, \tag{3.40}$$

$$\vec{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\,. \tag{3.41}$$

The constant normalization vector $\vec{n} = (n_1\,, n_2)^T$ will be determined later in section 3.3 from the consideration of global symmetry property. At present, however, it can be chosen arbitrarily. It is known that for a finite gap solution associated to a genus g algebraic curve the normalized Baker-Akhiezer vector has $g+1$ poles as a function of x. Therefore for an infinite gap solution of our interest it has infinite number of poles. We will denote the positions of these poles on the spectral curve by $\{\gamma_1, \gamma_2, \cdots\}$. Since the monodromy matrix Ω is constructed out of the string variables, through the relation (3.39) the positions of the poles γ_i on the spectral curve as well as the quasi-momentum at these poles $\hat{p}(\gamma_i)$ become dynamical variables. As described in [59,61] and derived slightly more rigorously in Appendix B, it turns out that the variables $\left(z(\gamma_i)\,, \frac{\sqrt{\lambda}}{4\pi i}\hat{p}(\gamma_i)\right)$, where z is the Zhukovsky

[13] Details of the limiting procedure will be discussed in section 3.2.3.

variable given in (3.36), form canonically conjugate pairs satisfying the following Poisson bracket relations

$$\frac{\sqrt{\lambda}}{4\pi i}\{z(\gamma_i)\,,\hat{p}(\gamma_j)\} = \delta_{ij}\,, \tag{3.42}$$

$$\{z(\gamma_i)\,,z(\gamma_j)\} = \{\hat{p}(\gamma_i)\,,\hat{p}(\gamma_j)\} = 0\,. \tag{3.43}$$

This shows that the filling fractions S_i defined previously provide the action variables of the system.

To construct the angle variables ϕ_i conjugate to S_i, we need to find the generating function $F(S_i\,,z(\gamma_i))$ which provides the canonical transformation from the pair $\left(z(\gamma_i)\,,\frac{\sqrt{\lambda}}{4\pi i}\hat{p}(\gamma_i)\right)$ to (ϕ_i, S_i). Such a function is defined by the following properties:

$$\frac{\partial F}{\partial z(\gamma_i)} = \frac{\sqrt{\lambda}}{4\pi i}\hat{p}(\gamma_i)\,, \tag{3.44}$$

$$\frac{\partial F}{\partial S_i} = \phi_i\,. \tag{3.45}$$

In the present context, the first equation should be viewed as the definition of F, while the second equation should be regarded as the definition of the angle variables ϕ_i. Therefore, to determine F, we need to integrate the first equation with S_i's fixed. Since the filling fractions are given by the integrals of $\hat{p}dz$ along various cycles on the spectral curve, fixing all the filling fractions is equivalent to fixing the functional form of $\hat{p}(x)$. Therefore, the integration can be performed as

$$F(S_i\,,z(\gamma_i)) = \frac{\sqrt{\lambda}}{4\pi i}\sum_i \int_{z(x_0)}^{z(\gamma_i)} \hat{p}(x')dz'\,. \tag{3.46}$$

The initial point of the integration x_0 on the spectral curve can be chosen arbitrarily. As it will become clear later in section 4, a change of x_0 can be absorbed by the change of overall normalization of the wave function. Similarly, a possible integration constant in F, which may depend only on S_i, can be ignored as it can also be absorbed in the normalization of the wave function.

Next we compute $\phi_i = \partial F/\partial S_i$. This requires changing the value of S_i with all the other filling fractions fixed. This is equivalent to adding to $\hat{p}dz$ a one-form whose period integral along a-cycles is nonvanishing only for a_i. Such a one-form should be proportional to a normalized holomorphic differential ω_i, which satisfies the following properties:

$$\oint_{a_j} \omega_i = \delta_{ij}\,, \qquad \oint_{C_s} \omega_i = -1\,. \tag{3.47}$$

Here C_s denotes a cycle which surrounds the singular cut. The integral over this cycle should cancel the contribution from the a_i cycle, since the integral along a large cycle surrounding all the branch cuts should vanish. Using such ω_i, the partial derivative $\partial F/\partial S_i$ can be expressed as

$$\phi_i = \frac{\partial F}{\partial S_i} = 2\pi \sum_j \int_{x_0}^{\gamma_j} \omega_i \,. \tag{3.48}$$

Here and hereafter, we regard ω_i as a differential in x. This is an appropriate generalization of the Abel map, which normally maps an algebraic curve to its Jacobian variety, for non-algebraic curves. When restricted to finite gap solutions, this expression exactly reproduces the definition of the Abel map.

We have now obtained an infinite set of action-angle variables, which satisfy the following canonical form of Poisson bracket relations:

$$\{\phi_i \,, S_j\} = \delta_{ij} \,, \qquad \{\phi_i \,, \phi_j\} = \{S_i \,, S_j\} = 0 \,. \tag{3.49}$$

However, there is an important caveat: Since the above construction is based purely on J^r, which is invariant under the left global transformation $\mathbb{X} \to V_L \mathbb{X}$, the angle variable conjugate to the right global charge S_0 cannot be obtained by this method[14]. To obtain such a variable, we need to make use of J^l. In an entirely similar manner, we can construct from J^l a set of angle variables $\tilde{\phi}_i$, which satisfy

$$\{\tilde{\phi}_i \,, S_j\} = \delta_{ij} \,, \qquad \{\tilde{\phi}_i \,, \tilde{\phi}_j\} = \{S_i \,, S_j\} = 0 \,. \tag{3.50}$$

The set $\{\tilde{\phi}_i\}$ contains the desired angle variable $\tilde{\phi}_0$ conjugate to S_0. However, it does not contain $\tilde{\phi}_\infty$, which is conjugate to S_∞. Therefore, to construct a complete set of angle variables, we must utilize the two individually incomplete sets, $(\phi_{i\neq 0,\infty} \,, \phi_\infty)$ and $(\tilde{\phi}_0 \,, \tilde{\phi}_{i\neq 0\,,\infty})$. A naïve guess would be to use $(\phi_{i\neq 0\,,\infty} \,, \phi_\infty)$ plus $\tilde{\phi}_0$. This, however, is not guaranteed to be correct since ϕ_i and $\tilde{\phi}_0$ do not commute in general. Nevertheless, we can use $(\tilde{\phi}_0 \,, \phi_{i\neq 0,\infty} \,, \phi_\infty)$ as if they constituted a complete set of angle variables, for the following reason. Suppose we find the "correct" angle variable ϕ_0 satisfying the following properties:

$$\{\phi_0 \,, S_0\} = 1 \,, \qquad \{\phi_0 \,, S_i\} = \{\phi_0 \,, \phi_i\} = 0 \qquad (i \neq 0) \,. \tag{3.51}$$

Then, from the Poisson bracket relations, we immediately see that the difference $\delta\phi_0 = \phi_0 - \tilde{\phi}_0$ commutes with all the action variables, namely $\{\delta\phi_0 \,, S_i\} = 0$ for all i. This

[14] In other words, the motion of such an angle variable is completely decoupled from the rest and cannot be seen from J^r.

means that it commutes with the worldsheet Hamiltonian, which is made up of the action variables S_i, and hence is conserved. Therefore $\delta\phi_0$ merely causes a constant shift of the angle variable and it can be absorbed in the normalization of the wave function. Thus, in practice, we can use $(\tilde{\phi}_0\,, \phi_{i\neq 0,\infty}\,, \phi_\infty)$ as a set of angle variables.

3.2.3 Reduction to finite gap cases

In this subsection, we explain how the method of construction of the action-angle variables developed above for infinite gap solutions can be applied to the case of the familiar finite gap solutions[15] associated with genus g algebraic curve, which are extensively studied in the literature [22, 59–61] and are believed to correspond to two point functions of various string states. We shall see that this procedure requires some careful considerations.

As is well-known, for a finite gap solution of genus g, there are $g+2$ non-vanishing filling fractions $(S_0\,, S_\infty; S_1\,, \cdots\,, S_g)$ and the associated normalized Baker-Akhiezer vector has $g+1$ dynamical poles. To obtain such a solution from an infinite gap solution, we must first set infinite number of filling fractions to zero, except for $(S_0\,, S_\infty; S_1\,, \cdots\,, S_g)$, by shrinking the corresponding cuts into node-like points. Through this degeneration process, the infinitely many poles of the Baker-Akhiezer vectors must somehow "disappear", leaving $g+1$ dynamical poles of the finite gap solutions. To understand what really happens, it is helpful to study similar degeneration limit for known finite gap solutions [59]. By closely analyzing the motion of the poles in such a degeneration limit, we find that actually the unwanted poles do not disappear. Instead, they cease to be dynamical[16]. These nondynamical poles cannot be seen if we use a solution with lower genus from the beginning. They can be seen only through the degeneration limit from a higher genus solution. This observation strongly suggests that, to obtain a complete set of action-angle variables, we should start from an infinite gap solution, construct the angle variables from infinitely many poles and then consider the limit of those angle variables. Carrying out this procedure, we can trace all the poles including nondynamical ones and obtain the following expression for the angle variables of a finite gap solution with genus g:

$$\phi_i = 2\pi \sum_{j=1}^{g+1} \int_{x_0}^{\gamma_j} \omega_i + 2\pi \sum_{J} \int_{x_0}^{\gamma_J} \omega_i\,. \tag{3.52}$$

[15] Although it is suggested in [62] that the finite gap method may be generalized to describe Wilson loops and correlation functions, here we use the term "finite gap solution" in the conventional sense. Namely, we mean the periodic solution constructed by the usual method [59] from an algebraic curve with finite genus, which can serve as the saddle point of a two point function when appropriately complexified. For a discussion on generalization of the finite gap method, also see section 3.2.4.

[16] A related discussion is given in [60].

Here, γ_j's denote the dynamical poles, while γ_J's signify the nondynamical ones.

Let us discuss the nature of the contributions from the nondynamical poles. A detailed argument on the motion of the poles given in Appendix E of [60] shows that nondynamical poles are trapped either at node-like points or at cusp-like points. Since such points are discretely placed on the spectral curve, the positions of the nondynamical poles γ_J do not change under any continuous deformations of the solution which keeps the spectral curve intact. In particular, they do not change under $SL(2,C)_L \times SL(2,C)_R$ global symmetry transformations. As we shall discuss later, the only necessary information for the evaluation of the correlation functions is the shift of angle variables under such global transformations. Thus, in practice, the second term in (3.52) gives the same constant contribution, which can be absorbed into the normalization of the wave function. Consequently, the angle variables for the finite gap solution can be effectively defined without the second term[17] as

$$\phi_i = 2\pi \sum_{j=1}^{g+1} \int_{x_0}^{\gamma_j} \omega_i \,. \tag{3.53}$$

This expression is quite convenient in practice since we do not have to consider the degeneration limits from the infinite gap solutions. Thus we will use (3.53) instead of (3.52) as the definition of angle variables for finite gap solutions when we evaluate the correlation functions later in section 4.

3.2.4 Structure of multi-pronged solutions and importance of local asymptotic behavior

The method of construction of the angle variables given above is for finite gap solutions, which serve as saddle point configurations for two point functions. Since we are interested in computing three and higher point functions as well, we must discuss how the method can be generalized to such cases.

Before giving the simple procedure, which turns out to require only the knowledge of the local behavior of the saddle point solution in the vicinity of each vertex insertion point, it is instructive to first clarify the difference of the analytic structures between two-prong and three-prong solutions[18] in the framework of the finite gap method.

[17] The expression (3.53) coincides with the one derived in [59, 61] for finite gap solutions. There it was derived within the finite dimensional subspace of the total phase space, appropriate for finite gap solutions with fixed genus. Our discussion in this section corroborates the result of [59, 61] from a more general point of view.

[18] The discussion to follow is applicable to higher-prong solutions as well.

Although a solution with three prongs is much more difficult to construct compared to the corresponding two-pronged solution, the behavior around each prong should be the same if it is generated by the same vertex operator. This implies that the spectral curve constructed from the local monodromy matrix should be the same as that of the two-point solution. Therefore the knowledge of the spectral curve alone cannot distinguish between two-point and three-point solutions.

What can distinguish between the two is the number of dynamical poles of the normalized Baker-Akhiezer vector. In the case of a finite gap solution of genus g relevant for a two-point function, there are $g+2$ non-vanishing filling fractions, which are dictated by the spectral curve, and $g+1$ dynamical poles of the normalized Baker-Akhiezer vector. The reconstruction formula then tells us that these two sets of data determine the (two-point) solution uniquely, up to a global symmetry transformation. What this implies is that for more general finite gap solutions, relevant for three point functions etc., the number of dynamical poles can be larger than $g+1$, while the number of branch cuts of finite length on the spectral curve remains to be $g+1$. This possibility has been overlooked until quite recently and is first utilized in [62] to reconstruct the solution which describes a correlation function of a circular Wilson loop and a half-BPS operator from the algebraic curve perspective[19]. The easiest way to obtain solutions with more than $g+1$ dynamical poles is to take the degeneration limit of the infinite gap solutions. Although only $g+1$ poles remained dynamical in the special degeneration limit considered in section 3.2.3, more general limits can be considered in which more than $g+1$ dynamical poles survive. In principle, it is even possible for the Baker-Akhiezer vector to have an infinite number of poles when the spectral curve has only a finite number of branch cuts. This phenomenon is demonstrated for a string in flat spacetime explicitly in Appendix H.

Despite the existence of important structural differences between two- and multi-pronged solutions as analyzed above, we now emphasize that as far as the evaluation of the angle variables needed to compute the contribution of the wave functions is concerned, only the local asymptotic behavior of the solution near the vertex insertion point suffices. This should indeed be the case because the vertex operator is defined locally and it produces the local source term for the equations of motion in the form

$$\partial\bar{\partial}X^{\mu} - (\partial X^{\nu}\bar{\partial}X_{\nu})X^{\mu} = \frac{\pi}{2\sqrt{\lambda}}\sum_i \frac{\delta \log V_i}{\delta X_{\mu}}\,. \tag{3.54}$$

Therefore possible local behavior around such a point is the same for two and higher

[19] In [62], the authors reconstructed the solution by requiring the existence of two distinct poles in the Baker-Akhiezer vector. Since the spectral curve of this solution has no branch cuts with finite length, this certainly goes beyond the ordinary finite gap construction.

point functions. In particular, in the case of LSGKP strings, this follows directly from the boundary condition we impose near the vertex insertion point, namely $\alpha \sim (1/4)\log p\bar{p}$. More precisely, as we shall describe in detail in the subsequent sections, the crucial information about the angle variables of the three point solutions needed for the evaluation of the wave functions can be extracted from the behavior of the angle variables for two point functions under suitable global symmetry transformations.

3.3 Global symmetry and evaluation of the wave function

Having discussed the method to construct a complete set of action-angle variables, we now explain how to actually evaluate the angle variables for a given finite gap solution and then compute the wave function in terms of the action-angle variables. In both of these processes, the global symmetry of the system will play the key role.

3.3.1 Shift of the angle variables under global symmetry transformations

In this subsection, we shall develop a general method for computing the angle variables for a given solution. We will describe the method for a general genus g finite gap solution in Euclidean AdS_3, *i.e.* within the $SL(2,C)$ principal chiral model we have been dealing with. However, as it will be clear, the method can be readily extended to any finite gap solution in more general spacetime. Moreover, it is relevant to the multi-point correlation functions, since the behavior of the saddle point solution in the vicinity of each vertex insertion point is the same as that of two-point solution.

Our method exploits the simple fact that from a certain "reference" finite gap solution of genus g one can generate solutions of the same type by the global $SL(2)_L \times SL(2)_R$ transformations (see figure 3.5). As we shall show, this procedure is sufficient to generate two point solutions of desired property from which we can compute the angle variables of our interest. As the reference solution, it is convenient to take the one for which the $SL(2)_L \times SL(2)_R$ charge matrices Q_L and Q_R are exactly diagonal, namely $Q_L = L\sigma_3/2, Q_R = -R\sigma_3/2$. (For the LSGKP case, the basic solution given in (2.9) has such a property.) We will denote the $SL(2)$ matrix form of such reference solution (as in (2.5)) by $\mathbb{X}^{\text{ref}}$. More general matrix solution $\mathbb{X}'$ can then be generated as

$$\mathbb{X}' = V_L \mathbb{X}^{\text{ref}} V_R \,, \qquad V_L \in SL(2)_L \,, \quad V_R \in SL(2)_R \,. \tag{3.55}$$

As already discussed in the previous section, for a genus g finite gap solution, there are $g+2$ non-vanishing action variables, including the charges R and L. We are interested in how the corresponding $g+2$ angle variables change as we go from $\mathbb{X}^{\text{ref}}$ to $\mathbb{X}'$.

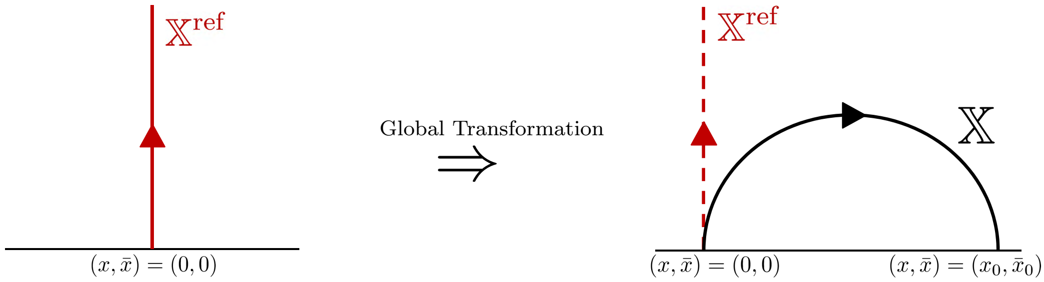

Figure 3.5: Schematic picture for producing a general two point solution $\mathbb{X}$ from a reference solution $\mathbb{X}^{\rm ref}$ by a global transformation.

Such angle variables are extracted from the positions of the poles of the suitably normalized eigenvector of the auxiliary linear problem. We will first employ the usual *left-invariant* Lax connection $J^r(x)$ to formulate the auxiliary linear problem. Then, the system is insensitive to the left charge L and its conjugate angle variable and hence only $g+1$ poles are visible. The information about the left charge and its conjugate will be extracted later from the similar analysis using the right-invariant Lax connection $J^l(x)$.

Let $\vec{\psi}^{\rm ref}(x;\tau,\sigma)$ be the eigenvector of the auxiliary linear problem for the reference solution $\mathbb{X}^{\rm ref}$. The normalized Baker-Akhiezer vector $\vec{h}^{\rm ref}(x;\tau)$ is proportional to $\vec{\psi}^{\rm ref}(x;\tau,\sigma=0)$ and satisfies the condition

$$\vec{n}\cdot\vec{h}^{\rm ref}(x;\tau)=n_1h_1+n_2h_2=1\,, \tag{3.56}$$

$$\vec{n}=\begin{pmatrix} n_1\\ n_2\end{pmatrix}\,,\quad \vec{h}^{\rm ref}=\begin{pmatrix} h_1\\ h_2\end{pmatrix}\,. \tag{3.57}$$

The choice of the constant normalization vector $\vec{n}$, extremely important for the construction of the proper wave function, will be discussed in detail in section 3.3.2. Until that point, $\vec{n}$ can be arbitrary. As the charge matrix Q_R is diagonal for our reference solution, the corresponding monodromy matrix $\Omega^{\rm ref}(x)$ behaves at $x\to\infty$ as

$$\Omega^{\rm ref}(x)=1-\frac{2\pi iR\sigma_3}{\sqrt{\lambda}x}+O(1/x^2)\,, \tag{3.58}$$

and hence the normalized eigenvector $\vec{h}^{\rm ref}(\infty^{\pm};\tau)$ at $x=\pm\infty$ take the form

$$\vec{h}^{\rm ref}(\infty^+;\tau)=\begin{pmatrix}\frac{1}{n_1}\\ 0\end{pmatrix}\propto\begin{pmatrix}1\\ 0\end{pmatrix}\,,\qquad \vec{h}^{\rm ref}(\infty^-;\tau)=\begin{pmatrix}0\\ \frac{1}{n_2}\end{pmatrix}\propto\begin{pmatrix}0\\ 1\end{pmatrix}\,. \tag{3.59}$$

Now, under the global transformation (3.55), $\vec{\psi}^{\rm ref}$ gets transformed into $V_R^{-1}\vec{\psi}^{\rm ref}$, where we parametrize V_R and its inverse as

$$V_R=\begin{pmatrix} v_{11} & v_{12}\\ v_{21} & v_{22}\end{pmatrix}\,,\qquad V_R^{-1}=\begin{pmatrix} v_{22} & -v_{12}\\ -v_{21} & v_{11}\end{pmatrix}\,. \tag{3.60}$$

The vector $\vec{h}^{\rm ref}$ gets rotated by the same matrix but in order to retain the normalization condition (3.57), we must rescale it appropriately. This gives

$$\vec{h}'(x;\tau) = \frac{1}{f(x;\tau)} V_R^{-1} \vec{h}^{\rm ref}(x;\tau)\,, \tag{3.61}$$

where the rescaling factor f is given by

$$\begin{aligned} f(x;\tau) &= \vec{n} \cdot \Big(V_R^{-1} \vec{h}^{\rm ref}(x;\tau) \Big) \\ &= n_1(v_{22}h_1 - v_{12}h_2) + n_2(-v_{21}h_1 + v_{11}h_2)\,. \end{aligned} \tag{3.62}$$

Hereafter, we shall suppress the τ-dependence as our focus will be on the behavior of functions and differentials on the spectral curve parametrized by x.

Let the positions of the poles of $\vec{h}^{\rm ref}$ and $\vec{h}'$ on the spectral curve be $\{\gamma_1, \gamma_2, \ldots, \gamma_{g+1}\}$ and $\{\gamma'_1, \gamma'_2, \ldots, \gamma'_{g+1}\}$ respectively[20]. Then, division by f must remove the poles $\{\gamma_1, \gamma_2, \ldots, \gamma_{g+1}\}$ while creating the new poles $\{\gamma'_1, \gamma'_2, \ldots, \gamma'_{g+1}\}$. In other words, the divisor of f is given by

$$(f) = \sum_{i=1}^{g+1} (\gamma'_i - \gamma_i)\,. \tag{3.63}$$

A natural meromorphic differential which encodes this information is

$$\varpi = d(\log f) = \frac{df}{f}\,. \tag{3.64}$$

From (3.63) ϖ must have poles at γ'_i and γ_i with residues 1 and -1 respectively. Besides, ϖ may have a holomorphic part, which can be written as a linear combination of the basic holomorphic differentials ω_i for $i = 1 \sim g$. They are assumed to be normalized in the usual way, namely $\int_{a_i} \omega_j = \delta_{ij}$, $\int_{b_i} \omega_j = \Pi_{ij}$, where Π_{ij} is the period matrix. To express this structure, let us introduce the basic abelian differential of the third kind ω_{PQ} characterized by the properties

$$\oint_P \omega_{PQ} = 1\,, \quad \oint_Q \omega_{PQ} = -1\,, \quad \oint_{a_i} \omega_{PQ} = 0\,. \tag{3.65}$$

Then, ϖ can be written as

$$\varpi = \sum_{i=1}^{g+1} \omega_{\gamma'_i \gamma_i} + \sum_{j=1}^{g} c_j \omega_j\,. \tag{3.66}$$

[20] As the number of poles in the normalized eigenfunction is dictated by the Riemann-Hurwitz theorem, it does not change under the global transformation. See [59] for details.

The expansion coefficients c_j are determined by the integrals of ϖ over the a_j-cycles. As ϖ is a differential of a logarithmic function, the possible contribution must be of the form

$$\int_{a_j} \varpi = 2\pi i m_j \,, \qquad m_j \in \mathbb{Z} \,. \tag{3.67}$$

This gives $c_j = 2\pi i m_j$. Next, consider the integrals of ϖ over the b_k-cycles. Again the possible structure is

$$\int_{b_k} \varpi = 2\pi i n_k \,, \qquad n_k \in \mathbb{Z} \,. \tag{3.68}$$

From (3.66) we then get

$$\sum_{i=1}^{g+1} \int_{b_k} \omega_{\gamma_i' \gamma_i} = 2\pi i n_k - 2\pi i \sum_{j=1}^{g} m_j \Pi_{jk} \,. \tag{3.69}$$

Now by using a variant of the Riemann bilinear identity[21], one can rewrite

$$\int_{b_k} \omega_{\gamma_i' \gamma_i} = 2\pi i \int_{\gamma_i}^{\gamma_i'} \omega_k \,. \tag{3.70}$$

Thus, (3.69) becomes

$$\sum_{i=1}^{g+1} \int_{\gamma_i}^{\gamma_i'} \omega_k = n_k - \sum_{j=1}^{g} m_j \Pi_{jk} \,. \tag{3.71}$$

Now note that n_k and m_j are integers which take discrete values. On the other hand, the LHS clearly vanishes continuously in the limit $\gamma_i \to \gamma_i'$. Hence, we should set $n_k = m_j = 0$ and conclude

$$\sum_{i=1}^{g+1} \int_{\gamma_i}^{\gamma_i'} \omega_k = 0 \,, \qquad k = 1 \sim g \,. \tag{3.72}$$

What this means is that the angle variables conjugate to the filling fractions $S_k, k = 1 \sim g$ do not change under the global transformation.

Therefore, the only angle variable left to be examined is the one associated with the differential $\omega_\infty \equiv \frac{1}{2\pi i} \omega_{\infty^+ \infty^-}$, namely the one conjugate to the charge R. We will denote it by ϕ_∞. This can be studied by considering the integral over the contour b_∞ running from ∞^- to ∞^+. Repeating essentially the same argument made for b_k, except for the evaluation of $\int_{b_\infty} \varpi$, we readily obtain

$$\int_{b_\infty} \varpi = \log \left(\frac{f(\infty^+)}{f(\infty^-)} \right) = 2\pi i \sum_{i=1}^{g+1} \int_{\gamma_i}^{\gamma_i'} \omega_\infty \,, \tag{3.73}$$

[21] See Corollary 2.42 of [59].

where we used an identity similar to (3.70), namely[22] $\int_{b_\infty} \omega_{\gamma_i' \gamma_i} = 2\pi i \int_{\gamma_i}^{\gamma_i'} \omega_\infty$. Since the RHS of (3.73) expresses the shift $\Delta\phi_\infty$ multiplied by i (see (3.53)), we have an important formula

$$\Delta\phi_\infty = -i \log\left(\frac{f(\infty^+)}{f(\infty^-)}\right) . \tag{3.74}$$

This can be recognized as the generalization of the formula given in Proposition 8.13 of [59], which was derived for the $U(1)_R$ part of the global transformation. Our master formula above is valid for an arbitrary global symmetry transformation.

In the present case, the formula can be made completely explicit by substituting the asymptotic behavior of $\vec{h}^{\rm ref}$ at $\infty^\pm$ given in (3.59) into the form of $f(x)$ in (3.62). This gives

$$\Delta\phi_\infty = -i \log\left(\frac{v_{22} - \frac{n_2}{n_1} v_{21}}{-\frac{n_1}{n_2} v_{12} + v_{11}}\right) . \tag{3.75}$$

Likewise, repeating exactly the same procedure with the *right-invariant* Lax connection, we can compute the shift of the angle variable $\tilde{\phi}_0$, conjugate to the $SL(2)_L$ charge L, which has been invisible in the left-invariant formalism. We obtain

$$\Delta\tilde{\phi}_0 = -i \log\left(\frac{\tilde{v}_{11} + \frac{\tilde{n}_2}{\tilde{n}_1} \tilde{v}_{21}}{\frac{\tilde{n}_1}{\tilde{n}_2} \tilde{v}_{12} + \tilde{v}_{22}}\right) , \tag{3.76}$$

where $\vec{\tilde{n}} = (\tilde{n}_1, \tilde{n}_2)^T$ is the normalization vector and $\tilde{v}_{ij}$ are the components of the $SL(2)_L$ transformation matrix V_L. The formulas (3.75) and (3.76) constitute the main result of this subsection.

3.3.2 Evaluation of the wave function from global symmetry

Once we have the information of the action-angle variables, the quantum wave function can be constructed in the manner

$$\Psi[\tilde{\phi}_0[\vec{\tilde{n}}], \phi_i[\vec{n}], \phi_\infty[\vec{n}]] \equiv \exp\left(iS_0 \tilde{\phi}_0[\vec{\tilde{n}}] + iS_\infty \phi_\infty[\vec{n}] + i \sum_i S_i \phi_i[\vec{n}]\right) . \tag{3.77}$$

To determine the wave function completely, however, we still have to specify the following quantities. One is the choice of the normalization vectors $\vec{n}$ and $\vec{\tilde{n}}$, on which the angle variables depend. We shall show below that this is related to the property of the wave function under the global symmetry transformations and there is a definite and proper

[22] See Proposition 2.43 of [59].

choice. The other is the choice of the origin of the angle variables (or equivalently, the choice of x_0 in (3.46)). This on the other hand is intimately related to the overall normalization of the wave function and will be determined when we evaluate the two point functions in the next section.

Let us study the property of the wave function under the global transformation. Under the transformation of the solution $\mathbb{X} \mapsto V_L \mathbb{X} V_R$, the currents transform as $J^r \to V_R^{-1} J^r V_R$ and $J^l \to V_L J^l V_L^{-1}$. In turn, the left and the right normalized Baker-Akhiezer vectors, which are the eigenvector of the auxiliary linear problem, transform as

$$\vec{h} \longmapsto V_R^{-1} \cdot \vec{h} \,, \qquad \vec{\tilde{h}} \longmapsto V_L \cdot \vec{\tilde{h}} \,. \tag{3.78}$$

Now in order to keep the normalization conditions, $\vec{n} \cdot \vec{h} = 1$ and $\vec{\tilde{n}} \cdot \vec{\tilde{h}} = 1$, $\vec{n}$ and $\vec{\tilde{n}}$ must transform in the opposite way, namely,

$$\vec{n} \longmapsto V_R^T \cdot \vec{n} \,, \qquad \vec{\tilde{n}} \longmapsto (V_L^T)^{-1} \cdot \vec{\tilde{n}} \,, \tag{3.79}$$

where the superscript T denotes transposition. This means that the wave function (3.77) transforms under the global symmetry transformation in the following way:

$$\Psi\left[\tilde{\phi}_0[\vec{\tilde{n}}], \phi_i[\vec{n}], \phi_\infty[\vec{n}]\right] \longmapsto \Psi\left[\tilde{\phi}_0[(V_L^T)^{-1} \cdot \vec{\tilde{n}}], \phi_i[V_R^T \cdot \vec{n}], \phi_\infty[V_R^T \cdot \vec{n}]\right] \,. \tag{3.80}$$

With this formula in mind, we shall now show that $\vec{n}$ and $\vec{\tilde{n}}$ are dictated by the properties required of the wave function (or the corresponding vertex operator) we wish to construct.

First we note that the wave function we want should describe, holographically, a certain single-trace operator in $\mathcal{N} = 4$ SYM and hence it should have the same transformation property as such an operator. For simplicity, let us consider the wave function corresponding to a conformal primary operator[23] of the gauge theory, carrying the dimension Δ and the spin S, inserted at the origin of the Minkowski space $R^{1,3}$, *i.e.* the boundary of AdS_5:

$$\Psi[\tilde{\phi}_0(\vec{\tilde{n}}), \phi_i(\vec{n}), \phi_\infty(\vec{n})] \qquad \longleftrightarrow \qquad \mathcal{O}^{\Delta\,,S}\,(x^\mu = 0) \,. \tag{3.81}$$

Now such a conformal primary operator at the origin of $R^{1,3}$ is completely invariant under the special conformal transformation. Therefore the corresponding wave function should

[23] Here we restrict our attention to conformal primary operators, namely highest weight operators. This is mainly because finite gap solutions including GKP strings are usually assumed to describe highest weight operators of the gauge theory (see e.g. [21]). It would be an interesting future problem either to find a way to describe non-highest weight operators by classical strings or to rigorously show that classical strings can only describe highest weight operators.

also be invariant under such a transformation given by

$$\Psi\left[\tilde{\phi}_0[\vec{\tilde{n}}], \phi_i[\vec{n}], \phi_\infty[\vec{n}]\right] \longmapsto \Psi\left[\tilde{\phi}_0[(V_L^{sc\,T})^{-1}\cdot\vec{\tilde{n}}], \phi_i[V_R^{sc\,T}\cdot\vec{n}], \phi_\infty[V_R^{sc\,T}\cdot\vec{n}]\right], \tag{3.82}$$

where $V^{sc}_{L,R}$ are the matrices given in Appendix C. For Ψ to be invariant, the angle variables should be invariant and hence the normalization vectors should be unchanged:

$$\vec{n} = V_R^{sc\,T}\cdot\vec{n}\,, \qquad \vec{\tilde{n}} = (V_L^{sc\,T})^{-1}\cdot\vec{\tilde{n}}\,. \tag{3.83}$$

From the form of $V^{sc}_{L,R}$ we readily find that the solutions are $\vec{n} = (1\,,0)^T$ and $\vec{\tilde{n}} = (0\,,1)^T$. One subtlety is that precisely for this special choice the number of dynamical poles of the Baker-Akhiezer vectors decreases by one. Thus to avoid this singular point, we should employ the "regularized" form

$$\vec{n} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}, \qquad \vec{\tilde{n}} = \begin{pmatrix} \tilde{\mu} \\ 1 \end{pmatrix}, \tag{3.84}$$

where μ and $\tilde{\mu}$ are "regularization parameters", to be set to zero at the end of the calculation.

It is important to note that the discussion above shows, incidentally, that our reference solution $\mathbb{X}^{\rm ref}$, which starts at the origin of the boundary at $\tau = -\infty$, precisely corresponds to such an insertion of $\mathcal{O}^{\Delta,S}(0)$. Recall that our reference solution is chosen so that the global charges Q_L and Q_R are diagonal. This corresponds to the state annihilated by the lowering operators of $SL(2)_L$ and $SL(2)_R$, which are indeed the special conformal generators from the point of view of the boundary theory.

So far we have only considered the case which corresponds to the insertion of the operator at the origin. When we consider the insertions at points other than the origin, the situation changes. For instance, while a conformal primary operator inserted at $x^\mu = 0$ is invariant under special conformal transformations, it is no longer so when inserted elsewhere on $R^{1,3}$. This is because the special conformal transformation acts also on the coordinates of $R^{1,3}$. Apparently, we must reanalyze the choice of the normalization vectors for such cases.

Fortunately, there is a way to avoid this complication. The key is again the use of the global transformation. The same type of operators inserted at different points are related to each other by a translation on the boundary. Thus the corresponding wave functions should also be likewise related. Furthermore, for saddle point approximation, evaluating the translated wave function on the original trajectory $\mathbb{X}_*$ is the same as evaluating the original (non-translated) wave function on the inversely translated trajectory

$(V_L^T)^{-1}\,\mathbb{X}_*\,(V_R^T)^{-1}$. This is expressed as (see figure 4.1)

$$\Psi\left[\tilde{\phi}_0[(V_L^T)^{-1}\cdot\vec{\tilde{n}}],\phi_i[V_R^T\cdot\vec{n}],\phi_\infty[V_R^T\cdot\vec{n}]\right]\Big|_{\mathbb{X}_*}=\Psi[\tilde{\phi}_0(\vec{\tilde{n}}),\phi_i(\vec{n}),\phi_\infty(\vec{n})]\Big|_{(V_L^T)^{-1}\mathbb{X}_*(V_R^T)^{-1}}\,. \tag{3.85}$$

It turns out to be much easier to consider the transformation of the trajectory than the transformation of the wave function. In this way, we only need to deal with the case corresponding to the insertion at the origin, for which the previous analysis is valid. This procedure will be carried out explicitly for the actual computation of the two point and the three point functions in subsequent sections.

Finally, let us return to the general formulas for the shift of the angle variables obtained in (3.75) and (3.76). Upon substituting the appropriate choice of the normalization vectors given in (3.84), these formulas become

$$\Delta\phi_\infty=-i\log\left(\frac{v_{22}-\mu v_{21}}{-\frac{v_{12}}{\mu}+v_{11}}\right)\,, \tag{3.86}$$

$$\Delta\tilde{\phi}_0=-i\log\left(\frac{\tilde{v}_{11}+\frac{\tilde{v}_{21}}{\tilde{\mu}}}{\tilde{\mu}\tilde{v}_{12}+\tilde{v}_{22}}\right)\,. \tag{3.87}$$

We will demonstrate that these simple and explicit formulas are extremely powerful in evaluating the wave functions for various correlation functions.

4 Computation of two point functions

In the previous section, we have developed a general method of computing the angle variables for the finite gap solution generated from the reference solution via a global symmetry transformation. The end results were the simple formulas (3.86) and (3.87), which do not depend on the details of the finite gap solution.

We shall now show that, by making use of these formulas, one can compute semi-classical two point functions for the cases where the saddle point configurations are finite gap solutions. After describing the essence of the method, which is quite universal, we will apply it to the general elliptic GKP strings, *i.e.* without taking the large spin limit. As it will become clear, the same reasoning can in fact be applied in the vicinity of each vertex insertion point of any higher-point function to compute the contribution of the vertex operators. This will be demonstrated explicitly for the three-point function of LSGKP strings in section 5.

4.1 General formula for two point functions

Although the logic of our method is quite general, for clarity let us consider the case of a two point function of the form $\langle V(x_0,\bar{x}_0\,;z_2)V(0,0\,;z_1)\rangle$ the semi-classical saddle of which is given by a finite gap solution. We will compute this correlation function by the use of the state-operator correspondence, without requiring the precise form of the vertex operators.

In the case of two point functions, the saddle point trajectory $\mathbb{X}$ is normally described in terms of the *global* cylinder coordinates (τ,σ) related to the plane coordinate z by [25]

$$e^{\tau+i\sigma}=\frac{z-z_1}{z-z_2}\,, \tag{4.1}$$

and it starts from $(0,0)$ at $\tau\simeq-\infty$ and ends on $(x_0,\bar{x}_0)$ at $\tau\simeq+\infty$, both on the boundary of the Euclidean AdS_3. On the other hand, the state-operator correspondence should be described using the *local* cylinder coordinates, namely (τ_1,σ_1) around z_1 and (τ_2,σ_2) around z_2, defined by

$$e^{\tau_1+i\sigma_1}=z-z_1\,,\qquad e^{\tau_2+i\sigma_2}=z-z_2\,. \tag{4.2}$$

Explicitly, around z_1 such a correspondence is described by

$$\int\mathcal{D}X\big|_{X=X_0\text{ at }\tau_1=\log\epsilon_1}V(0,0\,;z_1)e^{-S}=\Psi_1\left[X_0\right]\,, \tag{4.3}$$

where X_0 is the configuration prescribed on a small circle of radius ϵ_1 around z_1 and Ψ_1 is the wave function corresponding to the vertex operator $V(0,0\,;z_1)$. When the semi-classical approximation is adequate, we can replace the above path integral by the value at the saddle point configuration and the form of the correspondence simplifies to

$$V(0,0\,;z_1)\big|_{X_{\rm cl}}\exp\left[-S_{\rm cl}(\tau_1<\log\epsilon_1)\right]=\Psi_1\left[X_{\rm cl}\right]\big|_{\text{at }\tau_1=\log\epsilon_1}\,. \tag{4.4}$$

In a similar fashion, the vertex operator $V(x_0,\bar{x}_0\,;z_2)$ is related to the wave function Ψ_2, defined on a small circle of radius ϵ_2 around z_2.

To evaluate these wave functions, we need to reexpress the trajectory $\mathbb{X}$ parametrized by the global coordinates in terms of the local coordinates. In the vicinity of the vertex operators, the global coordinates are related to the local coordinates as

$$\tau+i\sigma\simeq\begin{cases}\log\left(\frac{z-z_1}{z_1-z_2}\right)=\tau_1+i\sigma_1-\log(z_1-z_2) & (z\sim z_1)\,,\\ \log\left(\frac{z_2-z_1}{z-z_2}\right)=-\tau_2-i\sigma_2+\log(z_2-z_1) & (z\sim z_2)\,.\end{cases} \tag{4.5}$$

In the discussion below, it will be quite important to keep track of which type of coordinates are being used in the description of the trajectory $\mathbb{X}$. Thus, to make this distinction clear, we shall employ the square bracket, like $\mathbb{X}[\tau_1, \sigma_1]$, when using the local coordinates, and the round bracket, like $\mathbb{X}(\tau, \sigma)$, for the use of the global coordinates. Using this notation, the trajectory near z_1 and z_2 can be expressed in two types of coordinates in the following way:

$$\mathbb{X}[\tau_1\,, \sigma_1] \simeq \mathbb{X}(\tau_1 - \tau_{12}\,, \sigma_1 - \sigma_{12}) \qquad (z \sim z_1)\,, \tag{4.6}$$

$$\mathbb{X}[\tau_2\,, \sigma_2] \simeq \mathbb{X}(-\tau_2 + \tau_{12}\,, -\sigma_2 + \sigma_{12} + \pi) \qquad (z \sim z_2)\,. \tag{4.7}$$

Here the quantities τ_{12} and σ_{12} are defined by

$$\tau_{12} \equiv \log |z_1 - z_2|\,, \qquad \sigma_{12} \equiv \frac{1}{2i} \log \left(\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \right)\,. \tag{4.8}$$

Then, including the contribution of the action evaluated in the global cylinder coordinate, the semi-classical two point function can be written as

$$\Psi_2\Big|_{\mathbb{X}[\tau_2\, =\, \log \epsilon_2]}\ \exp\left(-S\Big|_{\tau=\tau_i}^{\tau=\tau_f}\right)\ \Psi_1\Big|_{\mathbb{X}[\tau_1\, =\, \log \epsilon_1]}\,, \tag{4.9}$$

where the initial and the final global times are given by

$$\tau_i = \log \epsilon_1 - \log |z_1 - z_2|\,, \qquad \tau_f = -\log \epsilon_2 + \log |z_1 - z_2|\,. \tag{4.10}$$

Now let us discuss the actual evaluation of the wave functions. Semi-classical wave functions carrying definite values of the action variables can be constructed and evaluated easily if we employ the action-angle variables expressed in the local coordinates, as we describe below.

Consider first the "initial" wave function Ψ_1 corresponding to $V(0,0)$. To make the essence of the argument clear, let us focus on just one generic pair of action-angle variables, denoted by J and θ, without specifying which action-angle variables we are dealing with. Then, the wave function is given simply by $\Psi_1[\theta; \tau] = e^{iJ\theta - \mathcal{E}\tau}$, where $\mathcal{E}$ is the energy on the worldsheet[24]. Since $\mathcal{E}$ is a function only of J and the momentum $\mathcal{P}$ is of the form nJ with n an integer, as shown in [60], the angle variable θ depends linearly on the local coordinates when evaluated on the solution $\mathbb{X}$:

$$\theta\Big|_{\mathbb{X}[\tau_1\,, \sigma_1]} = -i\omega\tau_1 + n\sigma_1 + \theta\Big|_{\mathbb{X}[0,\,0]}\,. \tag{4.11}$$

[24] Although $\mathcal{E}$ vanishes for the relevant saddle point configuration, we will keep $\mathcal{E}$ throughout this subsection to demonstrate explicitly the equivalence of the Virasoro condition and the independence of the two-point function on the worldsheet coordinates.

Here, ω is the angular frequency given by $\partial\mathcal{E}/\partial J$ and the integer n is the mode number given in (3.34). (The presence of the factor of $-i$ in the first term on the right hand side is due to the use of the Euclidean worldsheet time τ.) By using the method described in subsection 3.2, it is possible, in principle, to evaluate (4.11) on a given classical solution. In practice, however, it is much more convenient to make use of the formula developed in subsection 3.3 and compute the angle variables for the solution relative to those for an appropriate reference solution. Since the two trajectories can be compared most easily when expressed in terms of the global cylinder coordinates, we rewrite $\theta\big|_{\mathbb{X}[\tau_1,\sigma_1]}$ using (4.6) as

$$\theta\big|_{\mathbb{X}[\tau_1\,,\sigma_1]} \overset{z\sim z_1}{\simeq} \theta\big|_{\mathbb{X}(\tau_1-\tau_{12}\,,\sigma_1-\sigma_{12})} = \varDelta\theta^{\mathbb{X}} + \theta\big|_{\mathbb{X}^{\rm ref}(\tau_1-\tau_{12}\,,\sigma_1-\sigma_{12})}\,, \tag{4.12}$$

where $\varDelta\theta^{\mathbb{X}}$ is defined as the difference

$$\varDelta\theta^{\mathbb{X}} \equiv \theta\big|_{\mathbb{X}(\tau_1-\tau_{12}\,,\sigma_1-\sigma_{12})} - \theta\big|_{\mathbb{X}^{\rm ref}(\tau_1-\tau_{12}\,,\sigma_1-\sigma_{12})}\,. \tag{4.13}$$

Note that $\varDelta\theta^{\mathbb{X}}$ is actually independent of τ_1 and σ_1 since both of the angle variables in (4.13) evolve linearly in τ_1 and σ_1. Thus we can define $\varDelta\theta^{\mathbb{X}}$ simply by $\theta\big|_{\mathbb{X}(\tau\,,\sigma)}-\theta\big|_{\mathbb{X}^{\rm ref}(\tau\,,\sigma)}$ at arbitrary values of the global coordinates. This observation allows us to compute $\varDelta\theta^{\mathbb{X}}$ by the formula developed in subsection 3.3, *i.e.* in terms of the parameters of the global transformation connecting $\mathbb{X}(\tau\,,\sigma)$ with $\mathbb{X}^{\rm ref}(\tau\,,\sigma)$. Using (4.12) Ψ_1 can be evaluated as

$$\begin{aligned}
\Psi_1\big|_{\mathbb{X}[\tau_1\,=\,\log\epsilon_1]} &= \exp\Big(iJ\theta\big|_{\mathbb{X}[\log\epsilon_1\,,0]} - \mathcal{E}\log\epsilon_1\Big) \\
&= \exp\Big(iJ\varDelta\theta^{\mathbb{X}} + iJ\theta\big|_{\mathbb{X}^{\rm ref}(\tau_i\,,-\sigma_{12})} - \mathcal{E}\log\epsilon_1\Big) \\
&= \exp\Big(iJ\varDelta\theta^{\mathbb{X}} + iJ\theta\big|_{\mathbb{X}^{\rm ref}(\tau_i\,,0)} - iJn\sigma_{12} - \mathcal{E}(\tau_i+\tau_{12})\Big) \\
&= \exp\big(iJ\varDelta\theta^{\mathbb{X}} - iJn\sigma_{12} - \mathcal{E}\tau_{12}\big)\,\Psi_1\big|_{\mathbb{X}^{\rm ref}(\tau_i)}\,,
\end{aligned} \tag{4.14}$$

where τ_i is the initial global time defined in (4.10).

Consider next the "final" wave function Ψ_2, which corresponds to the vertex operator $V(x_0,\bar{x}_0;z_2)$. One subtlety here is that the state-operator correspondence is made in the local cylinder coordinates $(\tau_2\,,\sigma_2)$, which, as seen in (4.5), run in directions opposite to the global cylinder coordinates $(\tau\,,\sigma)$. This means that we should actually evaluate the final wave function Ψ_2 on the *reversed solution* $\mathbb{X}(-\tau\,,-\sigma)$ to obtain the correct answer. With this in mind, we can compute the desired contribution in the following way. First we note that $V(x_0,\bar{x}_0)$ can be obtained from $V(0,0)$ by a translation by the vector $(x_0,\bar{x}_0)$ and denote it as $V(x_0,\bar{x}_0) = T_{x_0,\bar{x}_0}V(0,0)$. This means that the corresponding wave functions are also related by the translation as $\Psi_2 = T_{x_0\,,\bar{x}_0}\Psi_1$. Then, as can be seen from the

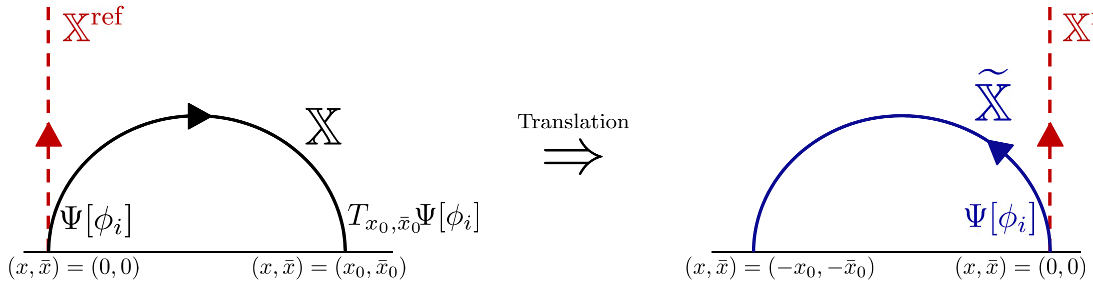

Figure 4.1: Schematic picture of how to evaluate the wave functions at two ends of the two point function, relative to their values for the reference solution $\mathbb{X}^{\text{ref}}$. The one at the origin is directly compared to $\mathbb{X}^{\text{ref}}$ (left figure). The one at $(x_0, \bar{x}_0)$ is evaluated by using the solution $\tilde{\mathbb{X}}$ obtained by translation and the subsequent reversal of the direction of τ and σ (right figure).

figure 4.1, evaluating Ψ_2 on the solution $\mathbb{X}$ is the same as evaluating Ψ_1 on the inversely translated solution, symbolically denoted as $T^{-1}_{x_0,\bar{x}_0}\mathbb{X}$. This is expressed more precisely as

$$(T_{x_0\,,\bar{x}_0}\Psi_1)\big|_{\mathbb{X}[\tau_2\,=\,\log\epsilon_2]} = \Psi_1\big|_{T^{-1}_{x_0,\bar{x}_0}\mathbb{X}[\tau_2\,=\,\log\epsilon_2]}\,. \tag{4.15}$$

Since Ψ_1 is given simply by $e^{iJ\theta-\mathcal{E}\tau}$, we can compute (4.15) by evaluating θ on $T^{-1}_{x_0,\bar{x}_0}\mathbb{X}$. To do this, we first rewrite $\theta\big|_{\mathbb{X}[\tau_2\,,\,\sigma_2]}$ in terms of the global coordinates and then reinterpret it as evaluated on the reversed solution. Explicitly,

$$\theta\big|_{T^{-1}_{x_0,\bar{x}_0}\mathbb{X}[\tau_2\,,\,\sigma_2]} \overset{z\sim z_2}{\simeq} \theta\big|_{T^{-1}_{x_0,\bar{x}_0}\mathbb{X}(-\tau_2+\tau_{12}\,,\,-\sigma_2+\sigma_{12}+\pi)} = \theta\big|_{\tilde{\mathbb{X}}(\tau_2-\tau_{12}\,,\,\sigma_2-\sigma_{12}-\pi)}\,, \tag{4.16}$$

where $\tilde{\mathbb{X}}$ is the reversed solution defined by

$$\tilde{\mathbb{X}}(\tau\,,\sigma) = T^{-1}_{x_0,\bar{x}_0}\mathbb{X}(-\tau\,,-\sigma)\,. \tag{4.17}$$

Note that, as shown in figure 4.1, this solution $\tilde{\mathbb{X}}$ starts from the origin just like the reference solution $\mathbb{X}^{\text{ref}}$. This allows us to further rewrite (4.16) as

$$\theta\big|_{\tilde{\mathbb{X}}(\tau_2-\tau_{12}\,,\,\sigma_2-\sigma_{12}-\pi)} = \varDelta\theta^{\tilde{\mathbb{X}}} + \theta\big|_{\mathbb{X}^{\text{ref}}(\tau_2-\tau_{12}\,,\,\sigma_2-\sigma_{12}-\pi)}\,, \tag{4.18}$$

where the difference $\varDelta\theta^{\tilde{\mathbb{X}}}$ is defined by

$$\varDelta\theta^{\tilde{\mathbb{X}}} = \theta\big|_{\tilde{\mathbb{X}}(\tau\,,\,\sigma)} - \theta\big|_{\mathbb{X}^{\text{ref}}(\tau\,,\,\sigma)}\,. \tag{4.19}$$

This quantity can be computed, just like $\varDelta\theta^{\mathbb{X}}$, by finding the global symmetry transformation which generates $\tilde{\mathbb{X}}(\tau\,,\sigma)$ from $\mathbb{X}^{\text{ref}}(\tau\,,\sigma)$. From (4.15), (4.16) and (4.18), Ψ_2 can

be evaluated as

$$\begin{aligned}\Psi_2\big|_{\mathbb{X}[\tau_2\,=\,\log\epsilon_2]} &= \exp\Big(iJ\theta\big|_{T_{x_0,\bar{x}_0}\mathbb{X}[\log\epsilon_2\,,\,0]} - \mathcal{E}\log\epsilon_2\Big)\\ &= \exp\Big(iJ\Delta\theta^{\tilde{\mathbb{X}}} + iJ\theta\big|_{\mathbb{X}^{\rm ref}(\tau_f\,,\,-\sigma_{12}-\pi)} - \mathcal{E}\log\epsilon_2\Big)\\ &= \exp\Big(iJ\Delta\theta^{\tilde{\mathbb{X}}} + iJ\theta\big|_{\mathbb{X}^{\rm ref}(\tau_f\,,\,0)} - iJn(\sigma_{12}+\pi) - \mathcal{E}(-\tau_f+\tau_{12})\Big)\\ &= \exp\Big(iJ\Delta\theta^{\tilde{\mathbb{X}}} - iJn(\sigma_{12}+\pi) - \mathcal{E}\tau_{12}\Big)\,\Psi_1\big|_{\mathbb{X}^{\rm ref}(-\tau_f)}\,, \end{aligned} \tag{4.20}$$

where τ_f is as given in (4.10).

Thus, combining the contributions form Ψ_1 and Ψ_2 given in (4.14) and (4.20) respectively, the net contribution of the wave functions can be written as

$$\Psi_1\,\Psi_2\big|_{\mathbb{X}} = (-1)^{Jn}\exp\Big(iJ(\Delta\theta^{\mathbb{X}} + \Delta\theta^{\tilde{\mathbb{X}}}) - 2(iJn\sigma_{12} + \mathcal{E}\tau_{12})\Big)\,\Psi_1\big|_{\mathbb{X}^{\rm ref}(\tau_i)}\Psi_1\big|_{\mathbb{X}^{\rm ref}(-\tau_f)}\,. \tag{4.21}$$

Recall that the sum of Jn's from all the angle variables is identified with the worldsheet momentum $\mathcal{P}$. Therefore, using the fact that the exponent of $\Psi_1\big|_{\mathbb{X}^{\rm ref}}$, given by $iJ\theta - \mathcal{E}\tau$, evolves linearly as $(J\omega - \epsilon)\tau$, (4.21) can be rewritten finally as

$$\begin{aligned}\Psi_1\,\Psi_2\big|_{\mathbb{X}} &= (-1)^{\mathcal{P}}\left(\Psi_1\big|_{\mathbb{X}^{\rm ref}(0)}\right)^2\exp\Big(iJ(\Delta\theta^{\mathbb{X}} + \Delta\theta^{\tilde{\mathbb{X}}}) - 2i\mathcal{P}\sigma_{12} - 2\mathcal{E}\tau_{12} - (J\omega-\mathcal{E})(\tau_f-\tau_i)\Big)\\ &= (-1)^{\mathcal{P}}\frac{\left(\Psi_1\big|_{\mathbb{X}^{\rm ref}(0)}\right)^2 e^{iJ(\Delta\theta^{\mathbb{X}}+\Delta\theta^{\tilde{\mathbb{X}}})}}{(z_1-z_2)^{\mathcal{E}+\mathcal{P}}(\bar{z}_1-\bar{z}_2)^{\mathcal{E}-\mathcal{P}}}\exp\left(-(J\omega-\mathcal{E})(\tau_f-\tau_i)\right)\,. \end{aligned} \tag{4.22}$$

Several important remarks are in order. First, the contribution from the reference solution $\left(\Psi_1\big|_{\mathbb{X}^{\rm ref}(0)}\right)^2$ controls the normalization of the two point function. Once we properly normalize the two point function by determining the value of $\Psi_1\big|_{\mathbb{X}^{\rm ref}(0)}$, the higher-point correlation functions, when computed with the same algorithm, will be properly normalized. Second, the dependence on the worldsheet coordinates z_1 and z_2 disappears, together with the sign factor $(-1)^{\mathcal{P}}$, if the solution satisfies the Virasoro conditions $\mathcal{E} = \mathcal{P} = 0$. This confirms the equivalence of the marginality of the vertex operator and the absence of the worldsheet coordinate dependence in the correlation functions discussed in [25]. Third, note that the form of the dependence on τ_i and τ_f is such that it precisely cancels with the contribution of the action $S[\theta]\big|_{\tau_i}^{\tau_f}$ constructed in terms of the action-angle variables[25] . Finally, the expression (4.22) tells us that the *spacetime* dependence of the two point function, the property of great interest, comes from the "phase shift" $e^{iJ(\Delta\theta^{\mathbb{X}}+\Delta\theta^{\tilde{\mathbb{X}}})}$.

Summarizing, we have established a simple procedure for computing the contribution of each vertex operator in any correlation functions. It consists of the following steps: Find

[25]In the case of the GKP string, this turns out to hold even when the action is evaluated on the solution expressed in terms of the embedding coordinates. For details, see section 4.2, Appendices D an E.

the behavior of the saddle point solution in the vicinity of the vertex operator in terms of the local cylinder time, translate it to the origin, and find the global transformation which produces this behavior from that of the reference solution $\mathbb{X}^{\rm ref}$. Then using the master formula we obtain the shift of the angle variables and hence the contribution of the wave function at that point.

4.2 Application to the case of elliptic GKP strings

To demonstrate the power and ease of our method developed above, let us apply it to the computation of the two point function for general elliptic GKP strings, *i.e.* without assuming the large spin limit. In this case, the reference solution is given by

$$\mathbb{X}^{\rm ref} = \begin{pmatrix} X_+^{\rm ref} & X^{\rm ref} \\ \bar{X}^{\rm ref} & X_-^{\rm ref} \end{pmatrix} = \begin{pmatrix} e^{-\theta(\tau)}\cosh\rho(\sigma) & e^{\phi(\tau)}\sinh\rho(\sigma) \\ e^{-\phi(\tau)}\sinh\rho(\sigma) & e^{\theta(\tau)}\cosh\rho(\sigma) \end{pmatrix}, \tag{4.23}$$

where $\theta(\tau) = \kappa\tau, \phi(\tau) = \omega\tau$ and $\sinh\rho(\sigma)$ can be expressed in terms of a Jacobi elliptic function, the detail of which will not be important for us for the essential part of the discussion below[26] . The only information we need in this subsection is that $\sinh\rho(\sigma)$ is a 2π-periodic odd function which vanishes at $\sigma = 0$. As was already explained in section 3.1, we should refer to the $\sigma = 0$ point of the string when we discuss the emission and the absorption of the string by the vertex operators. Then the reference solution above starts at $\tau = -\infty$ from the origin $(0,0)$ of the boundary of AdS_3 and reaches the horizon at $\tau = \infty$.

We now make a global transformation of the form

$$\mathbb{X} = \begin{pmatrix} X_+ & X \\ \bar{X} & X_- \end{pmatrix} = V_L \mathbb{X}^{\rm ref} V_R \tag{4.24}$$

such that the $\sigma = 0$ locus of the new solution $\mathbb{X}$ starts from $(0,0)$ and ends at $(x_0, \bar{x}_0)$, both on the boundary (see figure 3.5). Such a solution plays the role of the saddle point configuration for the two point function $\langle V(0,0)V(x_0,\bar{x}_0)\rangle$. The required conditions are

$$x\big|_{\sigma=0,\tau=-\infty} = \frac{X}{X_+}\bigg|_{\sigma=0,\tau=-\infty} = 0\,, \qquad \bar{x}\big|_{\sigma=0,\tau=-\infty} = \frac{\bar{X}}{X_+}\bigg|_{\sigma=0,\tau=-\infty} = 0\,, \tag{4.25}$$

$$x\big|_{\sigma=0,\tau=+\infty} = \frac{X}{X_+}\bigg|_{\sigma=0,\tau=+\infty} = x_0\,, \qquad \bar{x}\big|_{\sigma=0,\tau=+\infty} = \frac{\bar{X}}{X_+}\bigg|_{\sigma=0,\tau=+\infty} = \bar{x}_0\,. \tag{4.26}$$

The general form of V_L and V_R which achieve these conditions can be easily found to be

$$V_L = V_L^{(1)} \equiv \begin{pmatrix} a & \frac{1}{a\bar{x}_0} \\ 0 & \frac{1}{a} \end{pmatrix}, \qquad V_R = V_R^{(1)} \equiv \begin{pmatrix} a' & 0 \\ \frac{1}{a' x_0} & \frac{1}{a'} \end{pmatrix}, \tag{4.27}$$

[26] Nevertheless, when we elaborate on some details of the evaluation of the two point function in Appendix E, we will need the explicit form of the elliptic GKP string solution.

where a and a' are arbitrary parameters. This set of transformations consist of dilatation, rotation and special conformal transformation. It does not involve a translation because the $\tau = -\infty$ end of the string is tied to the origin $(0,0)$. Therefore we may use these forms in the general formulas given in (3.86) and (3.87) to obtain the following finite shifts of the angle variables:

$$\Delta\phi_\infty^{\mathbb{X}} = -i\log a'^2\,, \qquad \Delta\phi_0^{\mathbb{X}} = i\log a^2\,. \tag{4.28}$$

Next we compute the angle variables associated with the vertex operator $V(x_0,\bar{x}_0)$ evaluated on $\mathbb{X}$. According to the rule developed previously, all we have to do is find the global symmetry transformation which produces the configuration,

$$\tilde{\mathbb{X}}(\tau\,,\sigma) = T^{-1}_{x_0,\bar{x}_0}\mathbb{X}(-\tau\,,-\sigma) = T^{-1}_{x_0,\bar{x}_0}V_L^{(1)}\mathbb{X}^{\rm ref}(-\tau\,,-\sigma)V_R^{(1)}\,, \tag{4.29}$$

from $\mathbb{X}^{\rm ref}(\tau\,,\sigma)$. Under the reversal of the coordinates $(\tau\,,\sigma) \to (-\tau\,,-\sigma)$, the variables $\theta(\tau)$, $\phi(\tau)$ and $\sinh\rho(\sigma)$ in $\mathbb{X}^{\rm ref}$ flip sign, and this leads to the interchange $X_+^{\rm ref} \leftrightarrow X_-^{\rm ref}, X^{\rm ref} \leftrightarrow -\bar{X}^{\rm ref}$. This is effected by the global transformation of the form

$$\mathbb{X}^{\rm ref}(-\tau\,,-\sigma) = V_L^{(2)}\mathbb{X}^{\rm ref}(\tau\,,\sigma)V_R^{(2)}\,, \qquad V_L^{(2)} = i\sigma_2\,, \quad V_R^{(2)} = -i\sigma_2\,. \tag{4.30}$$

As for the translation[27] $T^{-1}_{x_0,\bar{x}_0}$, it is achieved by the matrices $V_L^{(3)} = V_L^{tr}(-\bar{x}_0)$ and $V_R^{(3)} = V_R^{tr}(-x_0)$, where $V_L^{tr}(\alpha)$ and $V_R^{tr}(\bar{\alpha})$ are given in (C.11). Altogether, we have

$$\tilde{\mathbb{X}}(\tau) = V_L\mathbb{X}^{\rm ref}V_R\,, \tag{4.31}$$

where

$$V_L = V_L^{(3)}V_L^{(1)}V_L^{(2)} = \begin{pmatrix} -\frac{1}{a\bar{x}_0} & a \\ 0 & -a\bar{x}_0 \end{pmatrix}, \tag{4.32}$$

$$V_R = V_R^{(2)}V_R^{(1)}V_R^{(3)} = \begin{pmatrix} -\frac{1}{a'x_0} & 0 \\ a' & -a'x_0 \end{pmatrix}. \tag{4.33}$$

Applying the general formulas for the shifts of the angle variables we obtain

$$\Delta\phi_0^{\tilde{\mathbb{X}}} = i\log\left(\frac{1}{a^2\bar{x}_0^2}\right) = -i\log(a^2\bar{x}_0^2)\,, \tag{4.34}$$

$$\Delta\phi_\infty^{\tilde{\mathbb{X}}} = -i\log\left(\frac{1}{a'^2x_0^2}\right) = i\log(a'^2x_0^2)\,. \tag{4.35}$$

[27] One might wonder at first sight that in the case of the spinning string the vertex operators $V(0,0)$ and $V(x_0,\bar{x}_0)$ carry opposite spin and hence they are not simply related by a translation. This suspicion is unfounded. In the case where the spinning string emanates and lands on the same plane, the direction of the spin, as seen from the respective vertex insertion point, is actually the same in the local coordinates and the vertex operators are related by a simple translation.

Finally, plugging the results (4.28), (4.34) and (4.35) into the formula (4.21), we obtain the contribution from the wave functions as

$$\begin{aligned}\Psi_1\Psi_2\big|_{\mathbb{X}} &= \exp\left(S_0\log(\bar{x}_0^2) - S_\infty \log(x_0^2)\right)\Psi_1\big|_{\mathbb{X}^{\rm ref}(\tau_i)}\Psi_1\big|_{\mathbb{X}^{\rm ref}(-\tau_f)} \\ &= \frac{\Psi_1\big|_{\mathbb{X}^{\rm ref}(\tau_i)}\Psi_1\big|_{\mathbb{X}^{\rm ref}(-\tau_f)}}{x_0^{(\Delta-S)}\bar{x}_0^{(\Delta+S)}}\,, \end{aligned} \tag{4.36}$$

where we substituted $S_\infty = -R/2 = -(\Delta - S)/2\,, S_0 = L/2 = (\Delta + S)/2$. As in (4.22), we can further express the reference wave functions at τ_i and $-\tau_f$ in terms of the value at $\tau = 0$. This gives

$$\Psi_1\big|_{\mathbb{X}^{\rm ref}(\tau_i)}\Psi_1\big|_{\mathbb{X}^{\rm ref}(-\tau_f)} = \exp\left(\left(\sum_i S_i\frac{\partial\mathcal{E}}{\partial S_i}\right)(\tau_f - \tau_i)\right)\left(\Psi_1\big|_{\mathbb{X}^{\rm ref}(0)}\right)^2 . \tag{4.37}$$

Note that the front factor is precisely of the form $\exp\left(S[\phi]\big|_{\tau_i}^{\tau_f}\right)$, where $S[\phi]$ denotes the action constructed in terms of the action-angle variables namely $S[\phi] = \int_{\tau_i}^{\tau_f} d\tau(\sum_i S_i\partial_\tau\phi_i - \mathcal{E})$, where $\mathcal{E}$ actually vanishes for our solution. Therefore this cancels precisely with the contribution of the action $\exp\left(-S[\phi]\big|_{\tau_i}^{\tau_f}\right)$ and we obtain

$$\Psi_1 e^{-S[\phi]}\Psi_2\big|_{\mathbb{X}} = \frac{\left(\Psi_1\big|_{\mathbb{X}^{\rm ref}(0)}\right)^2}{x_0^{(\Delta-S)}\bar{x}_0^{(\Delta+S)}} . \tag{4.38}$$

Upon setting $\Psi_1\big|_{\mathbb{X}^{\rm ref}(0)}$ to unity, we get the canonically normalized two point function for a spinning string. In Appendix E we will demonstrate explicitly that this result holds even when we use the action $S[X]$ constructed in the embedding coordinates. This shows that the possible extra contribution discussed in Appendix D can be ignored for the GKP strings.

5 Three point functions for LSGKP strings

Having developed a powerful method for evaluating the contribution of the vertex operators, we are now ready to complete our calculation of the three point functions for the LSGKP strings explicitly.

5.1 Contribution of the divergent part of the area

Let us begin with the evaluation of the divergent part of the area, which was given in (2.32) as $A_{div} = 4\int d^2z\sqrt{p\bar{p}}$. In order to see clearly how this contribution cancels exactly

with the one from the vertex operators, it is useful to express it as contour integrals using the Riemann bilinear identity. In this regard note that A_{div} can be written just like A_{reg} in (2.36), namely

$$A_{div} = 2i \int \sqrt{p(x)} dz \wedge \sqrt{\bar{p}(\bar{z})} d\bar{z} = 2i \int \lambda dz \wedge \omega \,, \tag{5.1}$$

where $\lambda = \sqrt{p(z)}$ as before and $\omega = \sqrt{\bar{p}(\bar{z})} d\bar{z}$, which is already a closed form. Thus we can make use of the generalized Riemann bilinear identity. Since the derivation, fully described in our previous paper, is lengthy, we do not repeat it here. The result is[28]

$$\begin{aligned} A_{div} = {} & i\left[2([\![C,d]\!]_\star - [\![d,C]\!]_\star) + \sum_i [\![C_i, C_i]\!]_\star + 2\sum_j ([\![C_j, \ell_j]\!]_\star - [\![\ell_j, C_j]\!]_\star)\right] \\ & - 2i \sum_i \oint_{C_i} \sqrt{\bar{p}} d\bar{z} \int_{z_i^*}^{z_i} \sqrt{p} dz' \,, \end{aligned} \tag{5.2}$$

where

$$[\![A, B]\!]_\star \equiv \int_A \sqrt{p} dz \int_B \sqrt{\bar{p}} d\bar{z} \,. \tag{5.3}$$

The contours C, d, C_i and ℓ_i are recalled in figure 2.1 for completeness but except for C_i we will not need them directly as we rewrite this formula into more convenient form below. The expression above is essentially identical to the general formula given in the equation (3.38) of [51] for A_{reg} for an N point function. There are, however, two differences which stem from the replacement $ud\bar{z} + vdz \to \sqrt{\bar{p}} d\bar{z}$. (For this reason we use the notation (5.3), which is slightly different from the one used in our previous paper [51].) First, a constant contribution $\pi(N-2)/12$ present in A_{reg}, which originates from the singularity of v from around the zeros of $p(z)$, is absent since here $\sqrt{\bar{p}}$ is not singular at those zeros. Contrarily, the contribution from a small circle around z_i of the form $\oint_{C_i} \omega$ vanished for $\omega = ud\bar{z} + vdz$ previously but it cannot be neglected here as $\omega = \sqrt{\bar{p}} d\bar{z}$ has poles at z_i.

Now we rewrite (5.2) into a more convenient form. By expressing the contours C and d_i in terms of basic "components" such as C_i, l_i and d, and using the property $[\![C_i, C_j]\!]_\star = [\![C_j, C_i]\!]_\star$ valid in the present case, the contours in (5.2) can be reassembled into the following form:

$$A_{div} = i\sum_i ([\![C_i, d_i]\!]_\star - [\![d_i, C_i]\!]_\star) + i\sum_i [\![C_i, C_i]\!]_\star - 2i\sum_i \oint_{C_i} \sqrt{\bar{p}} d\bar{z} \int_{z_i^*}^{z_i} \sqrt{p} dz' \,. \tag{5.4}$$

[28] In the published version of our previous paper [51], when reexpressing the quantity $\Lambda(z_0) - \Lambda(\hat{z}_0) \equiv \int^{z_0} \lambda dz - \int^{\hat{z}_0} \lambda dz$ in terms of a contour integral over λ, we inadvertently made an error and took the contour to be along d. When the analytic domain on which Λ is defined is correctly taken into account, the contour should be along $d + C$ instead. This revision introduces an additional contribution $[\![C, C]\!]_\star$ and the term $-\sum_{i\neq j} [\![C_i, C_j]\!]_\star$ in the formula (3.38) of [51] should be replaced by $+[\![C_i, C_i]\!]_\star$. Such a correction, shown in the erratum [51], is taken into account in deriving the expression shown in (5.2).

This turned out to be the most suitable form, since the terms in the first sum precisely cancel against the divergent contributions coming from the vertex operators, as we shall show in section 5.2. The double integrals in the last sum are easy to evaluate. Since these integrals are performed within the small vicinity of z_i, we may use the approximate form

$$\sqrt{p(z)} = \frac{\delta_i}{z - z_i} + \mathcal{O}((z - z_i)^0)\,, \tag{5.5}$$

and its complex conjugate for $\sqrt{\bar{p}(\bar{z})}$. Taking C_i to be a circle of infinitesimal radius ϵ_i around z_i, the double integral reduces to

$$\oint_{C_i} \sqrt{\bar{p}} d\bar{z} \int_{z_i^*}^{z_i} \sqrt{p} dz' = \oint_{C_i} d\bar{z} \left(\frac{\bar{\delta}_i \delta_i}{\bar{z} - \bar{z}_i} \ln \frac{z - z_i}{z_i^* - z_i} \right) + \mathcal{O}(\epsilon_i)\,. \tag{5.6}$$

This is easily evaluated by introducing the local angle parameter θ_i around z_i as $i\theta_i = \ln(z - z_i)/(z_i^* - z_i)$ and we get a finite result

$$\oint_{C_i} \sqrt{\bar{p}} d\bar{z} \int_{z_i^*}^{z_i} \sqrt{p} dz' = 2\pi^2 \bar{\delta}_i \delta_i \tag{5.7}$$

This term turns out to exactly cancel against the term $[\![C_i, C_i]\!]_\star$. Thus, we finally obtain the following expression for the divergent part of the area in terms of the contour integrals along d_i and C_i:

$$A_{div} = i \sum_i ([\![C_i, d_i]\!]_\star - [\![d_i, C_i]\!]_\star)\,. \tag{5.8}$$

We shall show in the next subsection that the same expression emerges from the calculation of the wave functions, with opposite sign, and they will cancel. Thus, we need not perform the complicated integrals along d_j appearing in (5.8).

5.2 Contribution of the vertex operators

The remaining task is to compute the contribution of the vertex operators, or the wave functions, and show that it combines with the divergent part of the area to produce a finite result. In what follows, we often refer to the results obtained in our previous paper, the essence of which is reviewed in section 2. For additional details, see [51].

Let ψ_a^L and ψ_a^R be the normalized solutions of the left and the right auxiliary problems where $p(z)$ is given by (2.42), which carries three double pole singularities of LSGKP type. If we can solve for ψ_a^L and ψ_a^R globally, we can obtain the saddle point configuration for the three point function of the LSGKP strings from the reconstruction formula (2.25),

which we display again:

$$\begin{pmatrix} X_+ & X \\ \bar{X} & X_- \end{pmatrix}_{a,\dot{a}} = (\psi_a^L, \psi_{\dot{a}}^R) \equiv \psi_{1,a}^L \psi_{\dot{1},\dot{a}}^R + \psi_{2,a}^L \psi_{\dot{2},\dot{a}}^R \, . \tag{5.9}$$

In practice, this is not possible. However, we can express the behavior of such saddle point solution near each of the singularities. First, near the singularity at z_i, we can obtain from (2.28) and (2.29) the basic "plus-minus" eigensolutions $i_\pm(\xi)$ of the auxiliary linear problem with the spectral parameter ξ, corresponding to the eigenvalues $e^{\pm i\hat{p}_i(\xi)}$ of the monodromy matrix M_i. These plus-minus eigenfunctions were defined only up to ξ-independent overall factor in our previous paper. However, to analyze the asymptotic behavior of the solution in detail we need to adopt a definite normalization. Here we choose the normalization such that the $i_\pm$ solutions asymptote to the following forms near the vertex insertion points:

$$1_\pm \sim \varphi_\pm^1 \, , \quad 2_\pm \sim \mp\varphi_\mp^2 \, , \quad 3_\pm \sim \varphi_\pm^3 \, , \tag{5.10}$$

$$\varphi_+^i(\xi) = \frac{1}{\sqrt{2}} e^{i\kappa_i(\xi^{-1}w - \xi\bar{w})} \begin{pmatrix} -ie^{(w-\bar{w})/8} \\ e^{-(w-\bar{w})/8} \end{pmatrix} , \tag{5.11}$$

$$\varphi_-^i(\xi) = \frac{1}{\sqrt{2}} e^{-i\kappa_i(\xi^{-1}w - \xi\bar{w})} \begin{pmatrix} -e^{(w-\bar{w})/8} \\ ie^{-(w-\bar{w})/8} \end{pmatrix} . \tag{5.12}$$

Here w is the local cylinder coordinate given by $w = \tau + i\sigma = \ln(z - z_i)$. Note that the definition of $2_\pm$ is slightly different from that of $1_\pm$ and $3_\pm$, reflecting the different behavior of $\sqrt{p(z)}$ near z_i on the first sheet, namely $\sqrt{p(z)} \simeq i\kappa_i/2(z - z_i)$ for $i = 1, 3$ and $\sqrt{p(z)} \simeq -i\kappa_i/2(z - z_i)$ for $i = 2$. $i_\pm(\xi)$ above are normalized under the $SL(2)$ invariant product as $\langle i_+, i_- \rangle = -\langle i_-, i_+ \rangle = 1$. From these ξ-dependent solutions, the solutions to the original left and right auxiliary linear problems are obtained as

$$i_\pm^L = i_\pm(\xi = 1) \, , \qquad i_\pm^R = e^{-i\pi/4} \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} i_\pm(\xi = i) \, . \tag{5.13}$$

Since they form a complete basis, we can expand the unknown global solutions $\psi_a^L, \psi_{\dot{a}}^R$ in the following way:

$$\psi_a^L = \langle \psi_a^L, i_-^L \rangle i_+^L - \langle \psi_a^L, i_+^L \rangle i_-^L \, , \qquad \psi_{\dot{a}}^R = \langle \psi_{\dot{a}}^R, i_-^R \rangle i_+^R - \langle \psi_{\dot{a}}^R, i_+^R \rangle i_-^R \, . \tag{5.14}$$

Although the coefficients, such as $\langle \psi_a^L, i_-^L \rangle$, cannot be computed directly, these are useful representations of $\psi^{L,R}$ in the vicinity of the insertion points. Plugging (5.14) in the formula (5.9) and using (5.12) and (5.13), we obtain the following solutions in the vicinity

of z_i:

$$
\begin{aligned}
X_+ &\simeq e^{\hat{\kappa}_i\tau}\beta_i^-(\alpha_i^+\mathrm{sh}_i-\alpha_i^-\mathrm{ch}_i)+e^{-\hat{\kappa}_i\tau}\beta_i^+(\alpha_i^-\mathrm{sh}_i-\alpha_i^+\mathrm{ch}_i)\,, && (5.15)\\
X_- &\simeq e^{\hat{\kappa}_i\tau}\overline{\beta}_i^-(\overline{\alpha}_i^+\mathrm{sh}_i-\overline{\alpha}_i^-\mathrm{ch}_i)+e^{-\hat{\kappa}_i\tau}\overline{\beta}_i^+(\overline{\alpha}_i^-\mathrm{sh}_i-\overline{\alpha}_i^+\mathrm{ch}_i)\,, && (5.16)\\
X &\simeq e^{\hat{\kappa}_i\tau}\overline{\beta}_i^-(\alpha_i^+\mathrm{sh}_i-\alpha_i^-\mathrm{ch}_i)+e^{-\hat{\kappa}_i\tau}\overline{\beta}_i^+(\alpha_i^-\mathrm{sh}_i-\alpha_i^+\mathrm{ch}_i)\,, && (5.17)\\
\bar{X} &\simeq e^{\hat{\kappa}_i\tau}\beta_i^-(\overline{\alpha}_i^+\mathrm{sh}_i-\overline{\alpha}_i^-\mathrm{ch}_i)+e^{-\hat{\kappa}_i\tau}\beta_i^+(\overline{\alpha}_i^-\mathrm{sh}_i-\overline{\alpha}_i^+\mathrm{ch}_i)\,. && (5.18)
\end{aligned}
$$

In the above formulas, to make the expressions compact, we have introduced the following abbreviations:

$$
\begin{aligned}
&\alpha_i^\pm\equiv\langle\psi_1^L,\hat{i}_\pm^L\rangle\,, \qquad \overline{\alpha}_i^\pm\equiv\langle\psi_2^L,\hat{i}_\pm^L\rangle\,, \quad \hat{i}_\pm^L\equiv\frac{1}{\sqrt{2}}(\pm i_+^L+i_-^L)\,, && (5.19)\\
&\beta_i^\pm\equiv\langle\psi_1^R,i_\pm^R\rangle\,, \qquad \overline{\beta}_i^\pm\equiv\langle\psi_2^R,i_\pm^R\rangle\,, && (5.20)\\
&\mathrm{ch}_i\equiv\cosh\hat{\kappa}_i\sigma\,, \qquad \mathrm{sh}_i\equiv\sinh\hat{\kappa}_i\sigma\,, && (5.21)\\
&\hat{\kappa}_{1,3}=\kappa_{1,3}\,, \qquad \hat{\kappa}_2=-\kappa_2\,. && (5.22)
\end{aligned}
$$

Note that $\hat{i}_\pm^L$ defined above satisfy the same normalization condition as $i_\pm^L$, that is, $\langle\hat{i}_+^L,\hat{i}_-^L\rangle=1$. The position $(x^{(i)},\bar{x}^{(i)})$ on the boundary from which the i-th string state emanates at the local cylinder time $\tau=-\infty$ can be expressed easily in terms of the coefficients α's and β's. Taking into account that $\hat{\kappa}_i=\kappa_i>0$ for $i=1,3$ while $\hat{\kappa}_2=-\kappa_2<0$, we obtain

$$
x^{(i)}=\left.\frac{X}{X_+}\right|_{\tau=-\infty,\sigma=0}=\begin{cases}\overline{\beta}_i^+/\beta_i^+ & \text{for } i=1,3\\ \overline{\beta}_i^-/\beta_i^- & \text{for } i=2\end{cases}\,, \tag{5.23}
$$

$$
\bar{x}^{(i)}=\left.\frac{\bar{X}}{X_+}\right|_{\tau=-\infty,\sigma=0}=\begin{cases}\overline{\alpha}_i^+/\alpha_i^+ & \text{for } i=1,3\\ \overline{\alpha}_i^-/\alpha_i^- & \text{for } i=2\end{cases}\,. \tag{5.24}
$$

We now show, by applying the method developed in the previous section, that the contribution of the wave functions to the three point function can be expressed in terms of the coefficients $\alpha_i^\pm$ and $\beta_i^\pm$. The first step is to translate the solution near z_i, given in (5.15)-(5.18), by the vector $(-x^{(i)},-\bar{x}^{(i)})$. This is effected by a transformation with the matrices $V_L^{tr}(-x^{(i)})$ and $V_R^{tr}(-\bar{x}^{(i)})$ given in Appendix C. For $i=1,3$, after the translation we get

$$
\begin{aligned}
\tilde{\mathbb{X}}=&V_L^{tr}(-\bar{x}^{(i)})\mathbb{X}(\tau)V_R^{tr}(-x^{(i)})\\
=&e^{-\kappa_i\tau}\begin{pmatrix}\beta_i^+(\alpha_i^-\mathrm{sh}_i-\alpha_i^+\mathrm{ch}_i) & 0\\ \frac{\beta_i^+}{\alpha_i^+}\mathrm{sh}_i & 0\end{pmatrix}+e^{\kappa_i\tau}\begin{pmatrix}\beta_i^-(\alpha_i^+\mathrm{sh}_i-\alpha_i^-\mathrm{ch}_i) & \frac{\alpha_i^+\mathrm{sh}_i-\alpha_i^-\mathrm{ch}_i}{\beta_i^+}\\ -\frac{\beta_i^-}{\alpha_i^+}\mathrm{ch}_i & -\frac{\mathrm{ch}_i}{\beta_i^+\alpha_i^+}\end{pmatrix}\,,
\end{aligned}
\tag{5.25}
$$

where the Schouten identity

$$
\langle\chi_1,\chi_2\rangle\langle\chi_3,\chi_4\rangle+\langle\chi_1,\chi_4\rangle\langle\chi_2,\chi_3\rangle+\langle\chi_1,\chi_3\rangle\langle\chi_4,\chi_2\rangle=0 \tag{5.26}
$$

has been utilized. This should be compared with the reference solution around z_i

$$\mathbb{X}^{\mathrm{ref}} = \begin{pmatrix} e^{-\kappa_i \tau} \cosh(\kappa_i \sigma) & e^{\kappa_i \tau} \sinh(\kappa_i \sigma) \\ e^{-\kappa_i \tau} \sinh(\kappa_i \sigma) & e^{\kappa_i \tau} \cosh(\kappa_i \sigma) \end{pmatrix} , \tag{5.27}$$

at $\tau \simeq -\infty$. Then we can see that $\tilde{\mathbb{X}}$ and $\mathbb{X}^{\mathrm{ref}}$ are related by the $SL(2)_L \times SL(2)_R$ transformation of the form

$$\tilde{\mathbb{X}} = V_L \mathbb{X}^{\mathrm{ref}} V_R \,, \tag{5.28}$$

$$V_L = \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} , \qquad V_R = \begin{pmatrix} a' & 0 \\ c' & \frac{1}{a'} \end{pmatrix} , \tag{5.29}$$

$$a = i\alpha_i^+ \,, \quad b = -i\alpha_i^- \,, \quad a' = i\beta_i^+ \,, \quad c' = -i\beta_i^- \,. \tag{5.30}$$

Applying the general formulas (3.86) and (3.87) for the shift of the angle variables, we immediately obtain

$$\Delta\phi_0^{(i)} = -i \log\left(\frac{1}{a^2}\right) = i \log\left(-(\alpha_i^+)^2\right) , \tag{5.31}$$

$$\Delta\phi_\infty^{(i)} = -i \log\left(a'^2\right) = -i \log\left(-(\beta_i^+)^2\right) . \tag{5.32}$$

We now repeat the similar analysis for $i = 2$ case. Looking at the form of the solutions (5.15)-(5.18), we see immediately that compared to the previous case we need to change the signs of the superscripts, such as $\alpha^\pm \to \alpha^\mp$, for all the coefficients. This change of the signs gives rise to additional (-1) factors when applying the Schouten identity. In this way we obtain

$$\tilde{\mathbb{X}} = e^{-\kappa_2 \tau} \begin{pmatrix} \beta_i^- (\alpha_i^+ \mathrm{sh}_i - \alpha_i^- \mathrm{ch}_i) & 0 \\ -\frac{\beta_i^-}{\alpha_i^-} \mathrm{sh}_i & 0 \end{pmatrix} + e^{\kappa_2 \tau} \begin{pmatrix} \beta_i^+ (\alpha_i^- \mathrm{sh}_i - \alpha_i^+ \mathrm{ch}_i) & \frac{\alpha_i^+ \mathrm{ch}_i - \alpha_i^- \mathrm{sh}_i}{\beta_i^-} \\ \frac{\beta_i^+}{\alpha_i^-} \mathrm{ch}_i & -\frac{\mathrm{ch}_i}{\beta_i^- \alpha_i^-} \end{pmatrix} . \tag{5.33}$$

The transformation from the reference solution is again given by (5.28) and (5.29), this time with the parameters

$$a = i\alpha_2^- \,, \quad b = i\alpha_2^+ \,, \quad a' = -i\beta_2^- \,, \quad c' = -i\beta_i^+ \,. \tag{5.34}$$

This gives the shift of the angle variables of the form

$$\Delta\phi_0^{(2)} = -i \log\left(\frac{1}{a^2}\right) = i \log\left(-(\alpha_2^-)^2\right) , \tag{5.35}$$

$$\Delta\phi_\infty^{(2)} = -i \log\left(a'^2\right) = -i \log\left(-(\beta_2^-)^2\right) . \tag{5.36}$$

Altogether, the contribution of the wave functions relative to that of the reference trajectory is given by

$$\exp\left(i \sum_{i=1}^{3} S_0^{(i)} \Delta\phi_0^{(i)} + S_\infty^{(i)} \Delta\phi_\infty^{(i)} \right) , \tag{5.37}$$

and is expressed solely in terms of the invariant products $\alpha_1^+, \alpha_2^-, \alpha_3^+, \beta_1^+, \beta_2^-, \beta_3^+$.

Thus our main task will be to evaluate these coefficients. As shown in (5.23) and (5.24), they are related to the positions of the vertex operators on the boundary. The quantities of essential significance for the correlation functions are rather the differences of these positions $x^{(i)} - x^{(j)}$ and $\bar{x}^{(i)} - \bar{x}^{(j)}$. Let us express them in terms of the above coefficients. For instance, $x^{(1)} - x^{(2)}$ is given by $(\overline{\beta}_1^+ \beta_2^- - \overline{\beta}_2^- \beta_1^+)/\beta_1^+ \beta_2^-$. By using the Schouten identity, we can identify the numerator as $-\langle 1_+^R, 2_-^R \rangle$ and hence the expression is simplified to $x^{(1)} - x^{(2)} = -\langle 1_+^R, 2_-^R \rangle / \beta_1^+ \beta_2^-$. In a similar fashion, one can express all the relative positions in terms of the $SL(2)$ invariant products $\alpha_1^+, \alpha_2^-, \alpha_3^+, \beta_1^+, \beta_2^-, \beta_3^+$ and $\langle 1_+^R, 2_-^R \rangle, \langle 1_+^R, 3_+^R \rangle, \langle 2_-^R, 3_+^R \rangle, \langle \hat{1}_+^L, \hat{2}_-^L \rangle, \langle \hat{1}_+^L, \hat{3}_+^L \rangle, \langle \hat{2}_-^L, \hat{3}_+^L \rangle$. These six relations in turn allow us to express[29] the six coefficients $\alpha_1^+, \alpha_2^-, \alpha_3^+, \beta_1^+, \beta_2^-, \beta_3^+$ in terms of the relative positions and the overlaps of the local eigenfunctions $\langle i_\pm^{R,L}, j_\pm^{R,L} \rangle$. The result is given by

$$(\alpha_1^+)^2 = \frac{-\langle \hat{1}_+^L, \hat{2}_-^L \rangle \langle \hat{3}_+^L, \hat{1}_+^L \rangle}{\langle \hat{2}_-^L, \hat{3}_+^L \rangle} \frac{(\bar{x}^{(2)} - \bar{x}^{(3)})}{(\bar{x}^{(1)} - \bar{x}^{(2)})(\bar{x}^{(3)} - \bar{x}^{(1)})}, \tag{5.38}$$

$$(\alpha_2^-)^2 = \frac{-\langle \hat{1}_+^L, \hat{2}_-^L \rangle \langle \hat{2}_-^L, \hat{3}_+^L \rangle}{\langle \hat{3}_+^L, \hat{1}_+^L \rangle} \frac{(\bar{x}^{(3)} - \bar{x}^{(1)})}{(\bar{x}^{(1)} - \bar{x}^{(2)})(\bar{x}^{(2)} - \bar{x}^{(3)})}, \tag{5.39}$$

$$(\alpha_3^+)^2 = \frac{-\langle \hat{3}_+^L, \hat{1}_+^L \rangle \langle \hat{2}_-^L, \hat{3}_+^L \rangle}{\langle \hat{1}_+^L, \hat{2}_-^L \rangle} \frac{(\bar{x}^{(1)} - \bar{x}^{(2)})}{(\bar{x}^{(3)} - \bar{x}^{(1)})(\bar{x}^{(2)} - \bar{x}^{(3)})}, \tag{5.40}$$

$$(\beta_1^+)^2 = \frac{-\langle 1_+^R, 2_-^R \rangle \langle 3_+^R, 1_+^R \rangle}{\langle 2_-^R, 3_+^R \rangle} \frac{(x^{(2)} - x^{(3)})}{(x^{(1)} - x^{(2)})(x^{(3)} - x^{(1)})}, \tag{5.41}$$

$$(\beta_2^-)^2 = \frac{-\langle 1_+^R, 2_-^R \rangle \langle 2_-^R, 3_+^R \rangle}{\langle 3_+^R, 1_+^R \rangle} \frac{(x^{(3)} - x^{(1)})}{(x^{(1)} - x^{(2)})(x^{(2)} - x^{(3)})}, \tag{5.42}$$

$$(\beta_2^+)^2 = \frac{-\langle 3_+^R, 1_+^R \rangle \langle 2_-^R, 3_+^R \rangle}{\langle 1_+^R, 2_-^R \rangle} \frac{(x^{(1)} - x^{(2)})}{(x^{(3)} - x^{(1)})(x^{(2)} - x^{(3)})}. \tag{5.43}$$

This is an important set of formulas. The left hand sides carry the information about how the global solutions $\{\psi_a^L, \psi_{\dot{a}}^R\}$ overlap with the local solutions near each insertion point. These quantities cannot be computed directly since we do not know the global solutions. This, however, is converted on the right hand side to the information about the relative positions and the overlaps of the eigenfuntions at different insertion points. Furthermore, when one makes proper identification of the states $i_\pm$ with the small solutions s_i around z_i, one recognizes that the ratios of their invariant products are computable using the results obtained in our previous paper.

First consider such a ratio present in β_1^+. Since $i_\pm^R$ type solutions are constructed from

[29] This is characteristic of the three point functions. For N-point functions, the number of relative positions is $N(N-1)$, while the number of overlaps of the type $\langle \psi_{\dot{1}}^R, i_\pm^R \rangle$ and $\langle \psi_1^L, \hat{i}_\pm^L \rangle$ (where the choice of $+$ or $-$ is unique for each i) is $2N$. These numbers match precisely for $N = 3$.

$i_{\pm}(\xi = i)$ by an $SL(2)$ transformation as in (5.13), the imaginary part of ξ is positive and the proper identification is $1_+ \propto s_1\,, 2_- \propto s_2\,, 3_+ \propto s_3$. To derive the precise relation between $i_\pm$ and s_i, we recall the form of the small solutions $s_i(\xi)$ given in [51]:

$$s_i(\xi) = \frac{e^{-i\pi/4}}{\sqrt{2}} \begin{pmatrix} e^{\alpha/2}\left(\pm\xi\partial S^+_{(i),odd}\right)^{-1/2} \\ e^{-\alpha/2}\left(\pm\xi\partial S^+_{(i),odd}\right)^{1/2}\left(1 \pm \frac{\bar{\partial} S_{(i),even}}{\bar{\partial} S^+_{(i),odd}}\right) \end{pmatrix} \exp\left[\pm S^+_{(i),odd}\right] . \tag{5.44}$$

Here $S^+_{(i),odd}$ is given by the power expansion in ξ as

$$S^+_{(i),odd} = \frac{1}{\xi}\int_{z_i^*} dz\sqrt{p} + \xi\left(\frac{1}{2}\int_{z_i^*} dz\{\frac{1}{\sqrt{p}}(\partial\hat{\alpha})^2 - \partial(\frac{1}{\sqrt{p}}\partial\hat{\alpha})\} + \int_{z_i^*} d\bar{z}\sqrt{\bar{p}}e^{-2\hat{\alpha}}\right) + \mathcal{O}(\xi^2)\,, \tag{5.45}$$

where z_i^* is a point close to z_i separated by ϵ_i. By analyzing the behavior of (5.44) around z_i and comparing it with (5.10), we find that for $\mathrm{Im}\,\xi > 0$,

$$1_+ = e^{i(\xi^{-1}-\xi)\kappa_1 \log\epsilon_1} s_1\,, \quad 2_- = e^{i(\xi^{-1}-\xi)\log\epsilon_2} s_2\,, \quad 3_+ = e^{i(\xi^{-1}-\xi)\log\epsilon_3} s_3\,. \tag{5.46}$$

Therefore we can reexpress the ratio in β_1^+ in terms of the small solutions as

$$\frac{\langle 1_+^R, 2_-^R\rangle\langle 3_+^R, 1_+^R\rangle}{\langle 2_-^R, 3_+^R\rangle} = e^{4\kappa_1\log\epsilon_1}\frac{\langle s_1, s_2\rangle\langle s_3, s_1\rangle}{\langle s_2, s_3\rangle}(\xi = i)\,. \tag{5.47}$$

We recognize that the ratio on the right hand side of (5.47) is nothing but the expression already encountered in (2.49) in the computation of the regularized area and hence can be evaluated: In [51], the part regular at $\xi = 0, \infty$ is determined by the use of the Wiener-Hopf decomposition, while the singular part is given simply by the first two terms of (2.49), *i.e.* integrals of $\sqrt{p}$ and $\sqrt{\bar{p}}$ along the contour d_1. The situation is similar for β_2^- and β_3^+.

Next we consider the ratios appearing in $\alpha_i^\pm$. We will evaluate the relevant invariant products by again relating them to those between the small solutions and then utilizing the results in our previous paper. However, there are two significant differences from the previous case for $\beta_i^\pm$'s. The first difference stems from the fact that the eigenfunctions $i_\pm^L$ in the ratios are obtained at $\xi = 1$. At this parameter point, there is no clear distinction between the small and the big solutions since the eigenfunctions asymptotically behave as $\exp[\pm\kappa_i(w - \bar{w})]$. A simple way to solve this problem is to take the limit from the region where $\mathrm{Im}\,\xi > 0$. Then we can identify the small solutions unambiguously as (5.46). Having specified the relation between $i_\pm$ and s_i, our next task is to evaluate the invariant products of $i_\pm$ by relating them to those of the small solutions. At this point, the second

difference arises because the relevant invariant products involve states other than s_i as well. For instance, $\langle \hat{1}_+^L, \hat{2}_-^L \rangle = \langle 1_+^L + 1_-^L, -2_+^L + 2_-^L \rangle/2$ contains 1_- and 2_+, which are not small solutions. Thus we cannot simply apply the results of our previous paper, where only the products between the small solutions were calculated. To determine the relevant invariant products, we take the following steps. First, the invariant products involving only the small solutions, $\langle 1_+^L, 2_-^L \rangle, \langle 2_-^L, 3_+^L \rangle$ and $\langle 3_+^L, 1_+^L \rangle$, can be computed just like the corresponding quantities in the R sector. The only difference here is that for $\xi = 1$ the prefactors $\exp\left(i(\xi^{-1} - \xi)\kappa_i \log \epsilon_i\right)$ in (5.46) are simply unity. Next recall that in [51] all the invariant products of $i_\pm$ are determined up to three unfixed constants, c, A_2 and A_3, which are connected to the normalization of $i_\pm$. By using the results for the three invariant products obtained above, we can now completely determine these constants. Then, this information in turn can be used to determine the rest of the invariant products. For details of the calculation, we refer the reader to Appendix F.

We are now ready to present the complete results for the contribution of the wave functions. Before showing the explicit details, however, it is useful to recall its general structure and discuss some of its features. It is of the form

$$\Psi_1 \Psi_2 \Psi_3 \big|_{\mathbb{X}} = \exp\left(i \sum_{i=1}^{3} S_0^{(i)} \mathit{\Delta}\phi_0^{(i)} + S_\infty^{(i)} \mathit{\Delta}\phi_\infty^{(i)} \right) \prod_{i=1}^{3} \Psi\big|_{\mathbb{X}^{\mathrm{ref}}(\log \epsilon_i)} . \tag{5.48}$$

As in the case of the two point function, the contribution from the reference wave functions $\Psi\big|_{\mathbb{X}^{\mathrm{ref}}(\log \epsilon_i)}$ contains divergences, which however will cancel against the ones from the action. On the other hand, the first term, which is of the form of the "phase shift" expressed in terms of the angle variables, is completely finite. Let us elaborate on this point. Recall that this term was evaluated by rewriting it in terms of the invariant products between the small solutions plus the prefactors shown in (5.47). As discussed in our previous work, the invariant products between the small solutions at general ξ are composed of the part regular at $\xi = 0, \infty$ and the singular part. As for the L sector, the singular part of the products between the small solutions vanishes by setting $\xi = 1$ and the prefactors are simply unity. Hence the phase shift coming from the L sector is manifestly finite. On the other hand, for the R sector, the singular part does not vanish. It is given essentially by the contour integral connecting two singularities and produces the divergences proportional to $\log \epsilon_i$. These divergences precisely cancel against the contribution from the prefactors given by $\exp(4\kappa_i \log \epsilon_i)$ in (5.47). Therefore, in total, the phase shift for the R sector is also finite. To present the results in a convenient form, however, we shall make the following slight rearrangement of the divergent contributions. From the result of Appendix E, we can calculate the ratio $\Psi\big|_{\mathbb{X}^{\mathrm{ref}}(\log \epsilon_i)} / \Psi\big|_{\mathbb{X}^{\mathrm{ref}}(0)}$. This turns

out to be exactly the inverse of the contribution from the prefactors in (5.47). Therefore it is convenient to cancel these $\log \epsilon_i$ divergences explicitly.

Combining altogether, the total contribution of the wave functions can be presented in the following form:

$$\begin{aligned}\Psi_1\Psi_2\Psi_3\big|_{\mathbb{X}} =&\frac{C_{\text{w.f.}}}{(x^1-x^2)^{\ell_1^-+\ell_2^--\ell_3^-}(x^2-x^3)^{\ell_2^-+\ell_3^--\ell_1^-}(x^3-x^1)^{\ell_3^-+\ell_1^--\ell_2^-}}\\&\times\frac{\left(\Psi\big|_{\mathbb{X}^{\text{ref}}(0)}\right)^3}{(\bar{x}^1-\bar{x}^2)^{\ell_1^++\ell_2^+-\ell_3^+}(\bar{x}^2-\bar{x}^3)^{\ell_2^++\ell_3^+-\ell_1^+}(\bar{x}^3-\bar{x}^1)^{\ell_3^++\ell_1^+-\ell_2^+}}\,.\end{aligned}\tag{5.49}$$

Here, the quantities $\ell_i^\pm$ are essentially the charges L and R carried by each of the vertex operator defined by

$$\ell_i^- = (\Delta^{(i)}-S^{(i)})/2\,,\qquad \ell_i^+\equiv(\Delta^{(i)}+S^{(i)})/2\,.\tag{5.50}$$

The log of the coefficient $C_{\text{w.f.}}$ is given by

$$\begin{aligned}\log C_{\text{w.f.}} =&H_-\left[h(x,\xi=i)\right]+H_+\left[h(x,\xi=1)\right]\\&+\frac{i\sqrt{\lambda}}{2}\sum_{j=1}^{3}\hat{\kappa}_j\left(\int_{d_j}\sqrt{p}dz-\int_{d_j}\sqrt{\bar{p}}d\bar{z}\right)+\sum_j\ell_j^+\log\tilde{c}\,,\end{aligned}\tag{5.51}$$

where the constant $\tilde{c}$ is of the form

$$\tilde{c}=1-\sqrt{\frac{\sinh(\pi(-\kappa_1+\kappa_2+\kappa_3))\sinh(\pi(\kappa_1-\kappa_2+\kappa_3))\sinh(\pi(\kappa_1+\kappa_2-\kappa_3))}{\sinh(\pi(\kappa_1+\kappa_2+\kappa_3))}}\,.\tag{5.52}$$

Finally the functions $H_\pm[*]$, which take a function as the argument, are defined as

$$\begin{aligned}H_\pm\left[f(x)\right]\equiv&\,2\sum_{j=1}^{3}\ell_j^\pm f(\kappa_j)-\left(\sum_{j=1}^{3}\ell_j^\pm\right)f(\sum_{j=1}^{3}\frac{\kappa_j}{2})-(-\ell_1^\pm+\ell_2^\pm+\ell_3^\pm)f(\frac{-\kappa_1+\kappa_2+\kappa_3}{2})\\&-(\ell_1^\pm-\ell_2^\pm+\ell_3^\pm)f(\frac{\kappa_1-\kappa_2+\kappa_3}{2})-(\ell_1^\pm+\ell_2^\pm-\ell_3^\pm)f(\frac{\kappa_1+\kappa_2-\kappa_3}{2})\,,\end{aligned}\tag{5.53}$$

and the actual argument functions $h(x,\xi=i)$ and $h(x,\xi=1)$ are defined in Appendix F. Concerning the contributions for $\log C_{\text{w.f.}}$, the first and the third terms are from the R sector whereas the second and the fourth terms are from the L sector.

Let us make a comment on some important features of (5.49) and (5.51). First, one should note that the integrals along d_j present in (5.51), such as $\hat{\kappa}_j\int_{d_j}\sqrt{p}dz$, exactly cancel the whole contribution from the divergent part of the area (5.8). Second, the form of the result (5.49) manifestly shows that the spacetime dependence of the three point function comes purely from the wave functions. This is as expected because the remaining contribution from the area of the worldsheet is invariant under the global symmetry transformations.

5.3 Final result for the three point functions and their properties

Now we combine all the contributions and obtain the final result for the three point function. Let us recall the various contributions we have calculated. The finite part of the action is calculated in our previous paper. Using the function H_-, it can be expressed as

$$\log C_{\text{area}} = -\frac{\sqrt{\lambda}}{2\pi} A_{\text{fin}} = -\frac{7\sqrt{\lambda}}{12} + H_- \left[K(x) \right] . \tag{5.54}$$

The divergent part of the action is studied in section 5.1 and the result for this contribution is given by (5.8). Lastly the contributions from the wave functions are calculated in the previous subsection and the results are given by (5.49) and (5.51).

When these contributions are put together, with the normalization condition $\Psi\big|_{\mathbb{X}^{\text{ref}}} = 1$ determined by the calculation of two point functions in section 4.2, the structure constant of the three point function is obtained as

$$\log C^{\text{LSGKP}} = -\frac{7\sqrt{\lambda}}{12} + H_-[\widetilde{K}(x)] + H_+[h(x, \xi = 1)] + \sum_j \ell_j^+ \log \tilde{c} \, , \tag{5.55}$$

where $\tilde{K}(x)$ and $h(x, \xi = 1)$ are given by

$$\tilde{K}(x) \equiv K(x) + h(x, \xi = i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \, \frac{\cosh 2\theta}{\cosh \theta} \log \left(1 - e^{-4\pi x \cosh \theta} \right) , \tag{5.56}$$

$$h(x, \xi = 1) = -\frac{1}{2} \log \left(1 - e^{-4\pi x} \right) . \tag{5.57}$$

It is intriguing to observe that a part of our final result (5.55) above has the same structure as the universal contribution from the AdS_2 part calculated in [50] for operators without Lorentz spins. To see this, we must actually subtract from the result of [50] "the regularized divergent contribution" of S^5 in order to extract the purely AdS_2 part. After such a subtraction, the result of [50] becomes essentially the same as $H_-[\widetilde{K}(x)]$, which in our case comes solely from the right-charge, *i.e.* $\Delta - S$, sector. From the technical point of view, this coincidence may not be so surprising, because both works are based on the same equation (2.14) with essentially the same boundary condition. A notable difference is, however, that while we explicitly take into account the contribution of the wave functions, the work of [50] does not deal directly with such contribution. In their approach, such effects appear to be indirectly taken into account by the way the worldsheet divergence is regularized using a cut-off in the target space. It would be important to understand the precise relation between the contribution from the wave functions and

the effect of such a cut-off in the target space, preferably from a physical point of view. On the other hand, there is no counterpart of our left-charge sector in [50], which comes from the presence of the spin in addition to the dilatation charge. In general, to get the complete dependence on the boundary coordinates for systems carrying quantum numbers other than the conformal dimension, one must consider the effect of the wave functions explicitly, as we have done.

Another property of our final result worth mentioning is its behavior under the limit where it is expected to reduce to a two point function. Specifically, consider the limit in which κ_3 becomes zero with $\kappa_2 \to \kappa_1$ at the same time. One might think that such a limit is beyond the domain of validity of our large charge approximation. However, if we set κ_3 to zero exactly, the saddle point trajectory should become the one for the two point function and hence C^{LSGKP} should reduce precisely to unity if all our normalization procedures have been correct. What actually happens is that if we take the limit $\kappa_3 \to 0$ naively in (5.55), we get $\log C^{\mathrm{LSGKP}} = -\sqrt{\lambda}/2$, which is different from the expected value 0. Let us trace the origin of this contribution. As discussed in our previous paper, this contribution comes from the following integral on the worldsheet

$$\int d^2z \partial\bar{\partial}\alpha \,. \tag{5.58}$$

As the integrand is a total derivative, it can be reexpressed as the boundary contour integral $\int(\partial\alpha dz - \bar{\partial}\alpha d\bar{z})$. The boundary in this case consists of the three small circles with radius ϵ_i around the vertex insertion points z_i and the large circle at infinity. Using the behavior $\exp(2\alpha) \sim \sqrt{p\bar{p}}$ in the vicinity of the insertion points, one finds that the integral around each insertion point yields π, which contributes to the structure constant the value $-\sqrt{\lambda}/2$. Since it is a constant independent of κ_i, the contribution from around z_3 remained in the naive limit $\kappa_3 \to 0$. Now the resolution of the puzzle is clear. The calculation sketched above is valid only when $\kappa_3 \gg \epsilon_3$. When we take κ_3 to zero, we must actually fix ϵ_3 in order for the singularity at z_3 to properly disappear. If we do this, then there is no contribution from around such a regular point and we obtain $\log C^{\mathrm{LSGKP}} = 0$ as desired.

6 Discussions

In this paper, in order to complete the calculation of the three point function for the LSGKP strings initiated in our previous work, we have developed a general method for evaluating the contribution of the vertex operators for semi-classical heavy string states,

which correspond to operators with large quantum numbers in $\mathcal{N} = 4$ SYM. In this method, we first construct the action-angle variables via the Sklyanin's method and then use them to evaluate the wave functions, which correspond to the vertex operators through the state-operator correspondence. We have tested this method for the two point functions and then applied it to the three point function for the LSGKP strings. Combined with the result of our previous work, we obtained the complete answer for the three point functions with the expected spacetime dependence.

Let us now mention several directions for further research.

Our present work has shown that the action-angle variables constructed *à la* Sklyanin can be extremely useful. This makes it important to construct such variables for a string in the full $AdS_5 \times S^5$ spacetime. For the bosonic sector, this should not be so difficult[30]. But the extension to the fermionic sector, which should be necessary for quantum consideration, may be nontrivial.

As concerns the use of the action angle variables, it should also be of great interest to reanalyze the calculation on the gauge theory side in terms of such variables, as already suggested in [11]. This will help us explore the connection between our results on the string theory side and the new integrable structures recently discovered on the SYM side [8–18].

In order to facilitate the comparison between the calculations of the three point functions on the string side with those on the gauge theory side, it is desirable to generalize the result of the gauge theory side from the $SU(2)$ sector to more general sectors, especially to the $SL(2)$ sector. An attempt in this direction is briefly discussed in the appendix of [11]. However, to obtain a general expression for three point functions of three long non-BPS operators, further detailed study will have to be performed. An alternative way to compare the results on two sides is to generalize our result for the string in Euclidean AdS_3 to the case in other spacetimes, in particular to a string in $AdS_2 \times S^3$, for which the corresponding analysis in the SYM theory is easier. Our general method of computing the contribution of the vertex operators can be applied to such cases as well. Since the action-angle variables for a string on $R \times S^3$ are already constructed, it should actually be straightforward for a string in $AdS_2 \times S^3$. However, the evaluation of the action may be technically involved, since the equations we obtain after the Pohlmeyer reduction are more complicated than for a string in Euclidean AdS_3. The task may be simplified if we could calculate the contribution from the action directly from the coset connection without resorting to the Pohlmeyer reduction. The direct evaluation of the action from the

[30] Generalization of the Sklyanin's method to the case of $GL(N)$ is already discussed in [58].

coset connection will also enable us to unify the calculation of the action and the vertex operators and it will help us uncover the structure underlying the entire calculation.

Another important project would be to evaluate the four point functions and understand their characteristic properties, in particular how the crossing symmetry is realized. Although qualitatively new features will come in, our method of calculation based on the global symmetry transformation should prove useful in such an investigation.

We are currently pursuing some of these directions and we hope to report our progress in the near future.

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Appendices

A: Derivation of the singularity structure of $\hat{p}(x)$

In this appendix, we give a derivation of the singularity structure of the quasi-momentum $\hat{p}(x)$ quoted in (3.25).

Consider first the singularity at $x = 1$ associated with that of the left-invariant Lax connection $J_z^r(x) = j_z/(1-x)$, where j_z is a 2×2 matrix given by $j_z = \mathbb{X}^{-1}\partial\mathbb{X}$. An important property of j_z for the string in Euclidean AdS_3 is that it must satisfy the Virasoro condition $\mathrm{Tr}\,[j_z j_z] = 0$. This does not of course mean that j_z vanishes. It means that j_z must be similar to a special Jordan block P, namely

$$j_z = uPu^{-1}\,, \qquad P \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{A.1}$$

Here u is a 2×2 matrix which depends on the worldsheet coordinate z. Note that P is nilpotent and so is j_z. To diagonalize the monodromy matrix to obtain $\hat{p}(x)$, it is convenient to go to a gauge where the connection is not degenerate. For this purpose we consider the gauge transformed Lax connection $\tilde{J}_z^r$ given by

$$\tilde{J}_z^r = u^{-1}J_z^r u + u^{-1}\partial u = \frac{P}{1-x} + A\,, \tag{A.2}$$

where $A \equiv u^{-1}\partial u$. Near $x = 1$, the eigenvalues $\lambda_\pm$ of $\tilde{J}^r_z$ are easily computed as

$$\lambda_\pm = \pm\sqrt{\frac{A_{21}}{1-x}} + \mathcal{O}((x-1)^0)\,. \tag{A.3}$$

A square root type singularity appeared[31]. The matrix element A_{21} can actually be expressed in terms of j_z. To see this, consider ∂j_z. Differentiating (A.1) we easily get $\partial j_z = (\partial u u^{-1})j_z - j_z(\partial u u^{-1})$. Then, taking the nilpotency of j_z into account, we get

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\,(\partial j_z \partial j_z) &= \mathrm{Tr}\,((\partial u u^{-1})j_z(\partial u u^{-1})j_z) \\ &= \mathrm{Tr}\,(\partial u P u^{-1}\partial u P u^{-1}) = \mathrm{Tr}\,(APAP) = (A_{21})^2\,. \end{aligned} \tag{A.4}$$

From (A.3) and (A.4), the quasi-momentum $\hat{p}(x)$ near $x = 1$ can be expressed as

$$\hat{p}(x) = \frac{2\pi\kappa_+}{\sqrt{1-x}} + O\left((x-1)^0\right)\,, \tag{A.5}$$

with

$$\kappa_+ = \frac{1}{2\pi i}\oint dz \left(\frac{1}{2}\mathrm{Tr}\;(\partial j_z \partial j_z)\right)^{1/4}\,. \tag{A.6}$$

For the behavior near $x = -1$, a similar analysis of $j_{\bar{z}}$ leads to

$$\hat{p}(x) = \frac{2\pi\kappa_-}{\sqrt{1+x}} + O\left((x+1)^0\right)\,, \tag{A.7}$$

$$\kappa_- = \frac{1}{2\pi i}\oint dz \left(\frac{1}{2}\mathrm{Tr}\;(\bar{\partial} j_{\bar{z}} \bar{\partial} j_{\bar{z}})\right)^{1/4}\,. \tag{A.8}$$

B: Remarks on the derivation of the canonical variables $(z(\gamma_i), \hat{p}(\gamma_i))$

The starting point of the construction of the action-angle variables described in section 3.2 was the existence of the canonical pair of variables $(z(\gamma_i), \hat{p}(\gamma_i))$. Although the derivation of the canonical Poisson brackets between these variables was given sometime ago in [61] for a string in $R \times S^3$, we would like to make some clarifying remarks on this derivation as applied to our system, namely a string in Euclidean AdS_3.

The first remark is that as we need not make the gauge-fixing of the target space time coordinate as was done in [61], we may use the Poisson bracket instead of the Dirac bracket.

The second remark is more important. In the derivation given in [61], the fact that the position of the poles of the Baker-Akhiezer function on the spectral curve, denoted

[31] This type of phenomenon commonly occurs when the dominant term is a Jordan block and the subdominant term is non-degenerate.

here[32] by γ_i, are dynamical variables made out of the string coordinates was not duly taken into account in some of the steps. Consequently, although the conclusions were correct, the derivation was somewhat misleading. Below we show that by treating the dynamical nature of γ_i's properly, with a change of the order of arguments, one can give a satisfactory derivation.

As in [61], it is convenient to perform a similarity transformation on the monodromy matrix Ω and its eigenvector $\vec{h}$ so that the normalization condition for the components of $\vec{h}$ becomes simple. As discussed in section 3.3.2, the appropriate normalization vector for our problem is $\vec{n} = (1, \mu)^T$ and the corresponding similarity transformation is given by

$$\tilde{\Omega}(x) \equiv \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \Omega(x) \begin{pmatrix} 1 & -\mu \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \tilde{\mathcal{A}}(x) & \tilde{\mathcal{B}}(x) \\ \tilde{\mathcal{C}}(x) & \tilde{\mathcal{D}}(x) \end{pmatrix} . \tag{B.1}$$

The components of $\tilde{\Omega}$ satisfy the same algebraic relations as the components of Ω, as in [61], which read

$$\{\tilde{\mathcal{B}}(x)\,, \tilde{\mathcal{B}}(x')\} = 0\,, \tag{B.2}$$

$$\begin{aligned}\{\tilde{\mathcal{A}}(x)\,, \tilde{\mathcal{B}}(x')\} = &\Big(\tilde{\mathcal{A}}(x)\tilde{\mathcal{B}}(x') + \tilde{\mathcal{A}}(x')\tilde{\mathcal{B}}(x)\Big)\, \hat{r}(x\,, x') \\ &+ \Big(\tilde{\mathcal{A}}(x)\tilde{\mathcal{B}}(x') + \tilde{\mathcal{D}}(x')\tilde{\mathcal{B}}(x)\Big)\, \hat{s}(x\,, x')\,, \end{aligned} \tag{B.3}$$

$$\{\tilde{\mathcal{A}}(x)\,, \tilde{\mathcal{A}}(x')\} = \Big(\tilde{\mathcal{B}}(x)\tilde{\mathcal{C}}(x') - \tilde{\mathcal{B}}(x')\tilde{\mathcal{C}}(x)\Big)\, \hat{s}(x\,, x')\,. \tag{B.4}$$

Here $\hat{r}(x, x')$ and $\hat{s}(x, x')$ are the coefficients of the Maillet's r-s matrices [63] given by

$$\hat{r}(x, x') \equiv -\frac{2\pi}{\sqrt{\lambda}} \frac{x^2 + x'^2 - 2x^2 x'^2}{(x - x')(1 - x^2)(1 - x'^2)}\,, \tag{B.5}$$

$$\hat{s}(x, x') \equiv -\frac{2\pi}{\sqrt{\lambda}} \frac{x + x'}{(1 - x^2)(1 - x'^2)}\,. \tag{B.6}$$

At the poles of the normalized Baker-Akhiezer vector, the components of $\widetilde{\Omega}$ satisfy the following relations[33]

$$\tilde{\mathcal{B}}(\gamma_i) = 0\,, \quad \tilde{\mathcal{D}}(\gamma_i) = \tilde{\mathcal{A}}(\gamma_i)^{-1} = e^{i\hat{p}(\gamma_i)}\,. \tag{B.7}$$

Now by studying these relations at the positions of the poles $x = \gamma_i, x' = \gamma_j$ one can obtain formulas necessary to derive the canonical brackets for $(z(\gamma_i), \hat{p}(\gamma_i))$. However, the limit such as $x \to \gamma_i$ must be taken with care. Such substitutions must be done *after* the computation of the Poisson brackets, since in obtaining the relations (B.2)-(B.4) the

[32] In [61] it was denoted by $x_{\hat{\gamma}_i}$. We use a simplified notation here.

[33] They follow simply from the eigenvalue equation, the normalization condition and the value of $\mathrm{Tr}\,\tilde{\Omega}$.

quantities x and x' have been treated as non-dynamical numbers. This was not duly taken into account in some of the procedures in [61]. For instance, the analysis in [61] starts with the naive substitutions of $x \to \gamma_i$ an $x' \to \gamma_j$ in (B.4), including those in the Poisson bracket on the left hand side. In this way the authors first derives the relation $\{\tilde{\mathcal{A}}(x)\,,\tilde{\mathcal{A}}(x')\} = 0$. This is not justified.

To derive the needed formulas properly, it is convenient to start from the analysis of (B.2) instead. Since $\tilde{\mathcal{B}}(x)$ has zeros at γ_i and γ_j $(i \neq j)$, it can be expressed as $\tilde{\mathcal{B}} = (x-\gamma_i)\mathcal{B}'$ or $\tilde{\mathcal{B}} = (x-\gamma_j)\mathcal{B}''$. The functions $\mathcal{B}'$ and $\mathcal{B}''$ are not known but what is important is that they have the properties $\mathcal{B}'(\gamma_i) \neq 0$ and $\mathcal{B}''(\gamma_j) \neq 0$. Then, (B.2) can be rewritten as

$$\begin{aligned} &(x-\gamma_i)(x'-\gamma_j)\{\mathcal{B}'(x)\,,\mathcal{B}''(x')\} - (x'-\gamma_j)\mathcal{B}'(x)\{\gamma_i\,,\mathcal{B}''(x')\} \\ &-(x-\gamma_i)\mathcal{B}''(x')\{\mathcal{B}'(x)\,,\gamma_j\} + \mathcal{B}'(x)\mathcal{B}''(x')\{\gamma_i\,,\gamma_j\} = 0\,. \end{aligned} \tag{B.8}$$

Now at this stage we can take the limit $x \to \gamma_i$ and $x' \to \gamma_j$. Then the first three terms vanish manifestly and from the last term we obtain the relation

$$\{\gamma_i\,,\gamma_j\} = 0\,. \tag{B.9}$$

Next step is to consider (B.3). Here again, we should substitute the expansion $\tilde{\mathcal{A}}(x) = \tilde{\mathcal{A}}(\gamma_i) + (x-\gamma_i)\mathcal{A}'(x)$ as well as the ones for $\mathcal{B}'$ and $\mathcal{B}''$. Then similarly to the case of (B.2), the limit $x \to \gamma_i$ and $x' \to \gamma_j$ can be taken easily and, making use of the relation (B.9), we can deduce the important relation

$$\{\tilde{\mathcal{A}}(\gamma_i)\,,\gamma_j\} = \frac{4\pi}{\sqrt{\lambda}}\tilde{\mathcal{A}}(\gamma_i)\frac{\gamma_i^2}{\gamma_i^2-1}\delta_{ij}\,. \tag{B.10}$$

Finally, similar calculation for (B.4) leads to

$$\{\tilde{\mathcal{A}}(\gamma_i)\,,\tilde{\mathcal{A}}(\gamma_j)\} = 0\,. \tag{B.11}$$

Both (B.9) and (B.10) are needed to obtain this result. The rest of the analysis is the same as in [61] and one proves that $(z(\gamma_i), \hat{p}(\gamma_i))$ constitute a canonical pair of variables.

C: $SL(2,C)_L \times SL(2,C)_R$ transformations as conformal transformations

In this appendix, we make a summary of the actions of the global symmetry transformations which play crucial roles in the main text. As we are interested in the saddle point configurations, which are in general complex, we will not impose any reality conditions

on the coordinates. Hence we will be concerned with $SO(4,C)$ which is isomorphic to $G = SL(2,C)_L \times SL(2,C)_R$. Below we consider these transformations from the point of view of conformal transformations.

Define the 2×2 matrix $\mathbb{X}$ and the action of G as in (2.5)-(2.8), namely,

$$\mathbb{X} \equiv \begin{pmatrix} X_+ & X \\ \bar{X} & X_- \end{pmatrix}, \qquad \det \mathbb{X} = 1\,, \tag{C.1}$$

$$X_+ = X_{-1} + X_4\,, \qquad X_- = X_{-1} - X_4\,, \tag{C.2}$$

$$X = X_1 + iX_2\,, \qquad \bar{X} = X_1 - iX_2\,, \tag{C.3}$$

$$\mathbb{X}' = V_L \mathbb{X} V_R\,, \qquad V_L \in SL(2)_L\,, \quad V_R \in SL(2)_R\,. \tag{C.4}$$

Then the basic transformations are identified as follows:

(1) Dilatation

$$X_+ \to \lambda X_+\,, \quad X_- \to \frac{1}{\lambda} X_-\,, \quad X, \bar{X}: \text{ invariant} \tag{C.5}$$

$$V_L^d(\lambda) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}, \quad V_R^d(\lambda) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}. \tag{C.6}$$

(2) Rotation

$$X \to \xi X\,, \quad \bar{X} \to \frac{1}{\xi} \bar{X}\,, \quad X_\pm: \text{ invariant} \tag{C.7}$$

$$V_L^r(\xi) = \begin{pmatrix} \sqrt{\xi} & 0 \\ 0 & \frac{1}{\sqrt{\xi}} \end{pmatrix}, \quad V_R^r(\xi) = \begin{pmatrix} \frac{1}{\sqrt{\xi}} & 0 \\ 0 & \sqrt{\xi} \end{pmatrix}. \tag{C.8}$$

(3) Translation

$$X \to X + \alpha X_+\,, \quad \bar{X} \to \bar{X} + \bar{\alpha} X_+\,, \quad X_+: \text{ invariant} \tag{C.9}$$

$$X_- \to X_- + \alpha \bar{X} + \bar{\alpha} X + \bar{\alpha}\alpha X_+\,, \tag{C.10}$$

$$V_L^{tr}(\bar{\alpha}) = \begin{pmatrix} 1 & 0 \\ \bar{\alpha} & 1 \end{pmatrix}, \quad V_R^{tr}(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}. \tag{C.11}$$

(4) Special conformal transformation

$$X \to X + \beta X_-\,, \quad \bar{X} \to \bar{X} + \bar{\beta} X_-\,, \quad X_-: \text{ invariant} \tag{C.12}$$

$$X_+ \to X_+ + \bar{\beta} X + \beta \bar{X} + \bar{\beta}\beta X_-\,, \tag{C.13}$$

$$V_L^{sc}(\beta) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad V_R^{sc}(\bar{\beta}) = \begin{pmatrix} 1 & 0 \\ \bar{\beta} & 1 \end{pmatrix}. \tag{C.14}$$

D: Effect of the canonical change of variables on the correlation function

Here we elaborate on the effect of the canonical change of variables on the calculation of the correlation functions mentioned in section 3.1. As is well-known, in classical mechanics the change of canonical variables from a set (q, p) to a new set (q', p') is effected by the transformation of the action of the form

$$S(q,p) = S'(q',p') + \int d\tau \frac{dF}{d\tau}\,, \tag{D.1}$$

where $F = F(q, p')$ is the generating function of the transformation giving p and q' in terms of (q, p') in the manner

$$\frac{\partial F}{\partial q} = p\,, \quad \frac{\partial F}{\partial p'} = q'\,. \tag{D.2}$$

Although the added term $\int d\tau (dF/d\tau)$ does not affect the classical equation of motion, it can produce a contribution when evaluated on a given trajectory if $F(\tau_f) \neq F(\tau_i)$:

$$S(q,p)\big|_{\tau=\tau_i}^{\tau=\tau_f} = S'(q',p')\big|_{\tau=\tau_i}^{\tau=\tau_f} + F(\tau_f) - F(\tau_i)\,. \tag{D.3}$$

The existence of such additional contribution can also be seen from a quantum-mechanical point of view. Consider the amplitude $\mathcal{A}$ for the transition from a state Ψ_1 to a state Ψ_2 in the path integral formulation. In terms of the variables (q, p), it is expressed as

$$\mathcal{A} = \int dq_i dq_f \int \mathcal{D}q\big|_{q(\tau_i)=q_i\,,\,q(\tau_f)=q_f} e^{-S[q]\big|_{\tau=\tau_i}^{\tau=\tau_f}} \bar{\Psi}_2(q_f)\Psi_1(q_i)\,, \tag{D.4}$$

where the wave functions are defined by $\Psi_1(q_i) \equiv \langle q_i|\Psi_1\rangle$, $\Psi_2(q_f) \equiv \langle q_f|\Psi_2\rangle$ and $\bar{\Psi}$ denotes the complex conjugate of Ψ. Suppose that the semiclassical approximation is valid and the path integral is dominated by a saddle point configuration q_*. Then, we have

$$\mathcal{A} = e^{-S[q_*]\big|_{\tau=\tau_i}^{\tau=\tau_f}} \bar{\Psi}_2(q_{f*})\Psi_1(q_{i*})\,. \tag{D.5}$$

Now if we express the same semi-classical amplitude in terms of a different set of canonical variables (q', p'), we likewise get

$$\mathcal{A} = e^{-S'[q'_*]\big|_{\tau=\tau_i}^{\tau=\tau_f}} \bar{\Psi}'_2(q'_{f*})\Psi'_1(q'_{i*})\,, \tag{D.6}$$

where the wave functions Ψ' are given by $\Psi'_1(q'_i) \equiv \langle q'_i|\Psi_1\rangle$, $\Psi'_2(q'_f) \equiv \langle q'_f|\Psi_2\rangle$. Comparing the two expressions (D.5) and (D.6), we find the following relation between the two actions evaluated on q_*:

$$S[q_*]\big|_{\tau=\tau_i}^{\tau=\tau_f} = S'[q'_*]\big|_{\tau=\tau_i}^{\tau=\tau_f} - \log\left(\Psi'_1(q'_{i*})/\Psi_1(q_{i*})\right) - \log\left(\bar{\Psi}'_2(q'_{f*})/\bar{\Psi}_1(q_{f*})\right)\,. \tag{D.7}$$

This can be simplified by using the relation between the two wave functions Ψ and Ψ', namely

$$\Psi'_1(q'_i) = \langle q'_i|\Psi_1\rangle = \int dq_i \langle q'_i|q_i\rangle\langle q_i|\Psi_1\rangle = \int dq_i \langle q'_i|q_i\rangle \Psi_1(q_i)\,. \tag{D.8}$$

In the semiclassical approximation, this relation is also dominated by the saddle point value and we may write

$$\Psi'_1(q'_{i*}) = \langle q'_{i*}|q_{i*}\rangle \Psi_1(q_{i*})\,. \tag{D.9}$$

Using the formula (D.9) and the similar one for q_f, the relation (D.7) can be reexpressed as

$$S[q_*]\big|_{\tau=\tau_i}^{\tau=\tau_f} = S'[q'_*]\big|_{\tau=\tau_i}^{\tau=\tau_f} - \log\langle q'_{i*}|q_{i*}\rangle + \log\langle q'_{f*}|q_{f*}\rangle\,. \tag{D.10}$$

Comparing (D.3) and (D.10), we find the useful identification

$$F(\tau_f) = \log\langle q'_{f*}|q_{f*}\rangle\,, \quad F(\tau_i) = \log\langle q'_{i*}|q_{i*}\rangle\,. \tag{D.11}$$

The discussion above clearly shows that, when we evaluate the action and the wave functions in terms of different canonical variables, we must add appropriate contribution of the form $\log\langle q'|q\rangle$ to $\log\mathcal{A}$ in order to obtain the correct amplitude. Explicitly,

$$\begin{aligned}\log\mathcal{A} = &- S[q_*]\big|_{\tau=\tau_i}^{\tau=\tau_f} + \log\Psi'_1(q'_{i*}) + \log\bar{\Psi}'_2(q'_{f*}) \\ &+ \log\langle q'_{f*}|q_{f*}\rangle - \log\langle q'_{i*}|q_{i*}\rangle\,. \end{aligned} \tag{D.12}$$

This remark also applies to our calculation of the correlation functions in this paper, where we make the change of variables from the original embedding coordinates $X_\mu(\sigma)$ to the angle variables ϕ_i when computing the wave functions. Therefore, in principle, we need to add additional contributions of the form $\log\langle X(\sigma)|\phi_i\rangle$. In the case of the correlation functions for the GKP strings, however, we will show in Appendix E that these terms actually do not contribute to the final result. Arguments for more general cases are yet to be developed.

E: Some details of the two point function of the elliptic GKP strings

In this appendix, we present some details of the evaluation of the two point function of the elliptic GKP strings left out in the discussion of section 4.2. In particular, we will explicitly compute the contributions of the action $S[X]\big|_{\tau=\tau_i}^{\tau=\tau_f}$ and of the reference

wave functions $\Psi_{\mathbb{X}^{\rm ref}(\tau_i)}$ and $\Psi_{\mathbb{X}^{\rm ref}(-\tau_f)}$. We will show that the dependence on τ_i and τ_f exactly cancel among these contributions and argue that the extra contribution of the type $\log\langle X(\sigma)|\phi_i\rangle$ discussed in Appendix D does not affect the two point function in the case of the GKP strings.

Let us recall the reference solution for the elliptic GKP strings, shown previously in (4.23). It is of the form

$$\mathbb{X}^{\rm ref}(\tau) = \begin{pmatrix} e^{-\kappa\tau}\cosh\rho(\sigma) & e^{\omega\tau}\sinh\rho(\sigma) \\ e^{-\omega\tau}\sinh\rho(\sigma) & e^{\kappa\tau}\sinh\rho(\sigma) \end{pmatrix}, \tag{E.1}$$

where κ, ω, $\cosh\rho(\sigma)$ and $\sinh\rho(\sigma)$ are given by

$$\kappa \equiv \frac{2k}{\pi}\mathcal{K}(k^2)\,, \quad \omega \equiv \frac{2}{\pi}\mathcal{K}(k^2)\,, \tag{E.2}$$

$$\cosh\rho(\sigma) \equiv \frac{1}{\sqrt{1-k^2}}{\rm dn}\left(\omega(\sigma+\pi/2)\right)\,, \quad \sinh\rho(\sigma) \equiv \frac{k}{\sqrt{1-k^2}}{\rm cn}\left(\omega(\sigma+\pi/2)\right)\,. \tag{E.3}$$

Here $k(\leq 1)$ is the elliptic modulus, $\mathcal{K}(k^2)$ is the the complete elliptic integral of the first kind and ${\rm dn}(u)$ and ${\rm cn}(u)$ are the Jacobi elliptic functions.

For this solution the action can be easily calculated as

$$S[X]\Big|_{\tau=\tau_i}^{\tau=\tau_f} = \frac{\sqrt{\lambda}}{\pi}\int d^2z\partial\vec{X}\bar{\partial}\vec{X} = \frac{\sqrt{\lambda}}{2\pi}\kappa^2(\tau_f-\tau_i)\int d\sigma\,{\rm sn}^2\left(\omega(\sigma+\pi/2)\right)\,. \tag{E.4}$$

Explicit evaluation of the integral, though not difficult, is unnecessary as it will be seen to be canceled by the contribution of the wave functions.

As for the evaluation of the wave functions $\Psi\big|_{\mathbb{X}^{\rm ref}(\tau_i)}$ and $\Psi\big|_{\mathbb{X}^{\rm ref}(-\tau_f)}$, it suffices to express them in terms of the value at $\tau=0$, namely $\Psi\big|_{\mathbb{X}^{\rm ref}(0)}$. This can be achieved by the use of the universal formula derived in section 3.3 since the time evolution of the reference solution can be effected by a global symmetry transformation. It consists of a dilatation and a rotation in the form

$$\mathbb{X}^{\rm ref}(\tau) = \begin{pmatrix} e^{(\omega-\kappa)\tau/2} & 0 \\ 0 & e^{-(\omega-\kappa)\tau/2} \end{pmatrix}\mathbb{X}^{\rm ref}(0)\begin{pmatrix} e^{-(\omega+\kappa)\tau/2} & 0 \\ 0 & e^{(\omega+\kappa)\tau/2} \end{pmatrix}. \tag{E.5}$$

This allows us to compute the difference of the angle variables as

$$\mathit{\Delta}\phi_0 = \phi_\infty\big|_{\mathbb{X}^{\rm ref}(\tau)} - \phi_\infty\big|_{\mathbb{X}^{\rm ref}(0)} = -i(\omega-\kappa)\tau\,, \tag{E.6}$$

$$\mathit{\Delta}\phi_\infty = \phi_\infty\big|_{\mathbb{X}^{\rm ref}(\tau)} - \phi_\infty\big|_{\mathbb{X}^{\rm ref}(0)} = -i(\omega+\kappa)\tau\,. \tag{E.7}$$

On the other hand, the action variables S_0 and S_∞ for the elliptic GKP string solution are given by $S_0 = (\Delta+S)/2$ and $S_\infty = -(\Delta-S)/2$, where the global charges Δ and S are expressed as

$$\Delta = \frac{\sqrt{\lambda}}{2\pi}\kappa\int_0^{2\pi} d\sigma\cosh^2\rho(\sigma)\,, \qquad S = \frac{\sqrt{\lambda}}{2\pi}\omega\int_0^{2\pi} d\sigma\sinh^2\rho(\sigma)\,. \tag{E.8}$$

From the expressions (E.6)-(E.8) and with the use of some basic identities for the elliptic functions, we can express the wave functions at τ_i and $-\tau_f$ in terms of the one at $\tau = 0$ as

$$\begin{aligned}
\Psi\big|_{\mathbb{X}^{\rm ref}(\tau_i)} &= \exp\left(iS_0\varDelta\phi_0 + iS_\infty\varDelta\phi_\infty\right)\Psi\big|_{\mathbb{X}^{\rm ref}(0)} \\
&= \exp\left(-\frac{\sqrt{\lambda}\kappa^2\tau_i}{2\pi}\int_0^{2\pi} d\sigma\, \mathrm{sn}^2(\omega(\sigma+\pi/2))\right)\Psi\big|_{\mathbb{X}^{\rm ref}(0)}\,, \qquad \text{(E.9)}\\
\Psi\big|_{\mathbb{X}^{\rm ref}(-\tau_f)} &= \exp\left(iS_0\varDelta\phi_0 + iS_\infty\varDelta\phi_\infty\right)\Psi_{\mathbb{X}^{\rm ref}(0)} \\
&= \exp\left(\frac{\sqrt{\lambda}\kappa^2\tau_f}{2\pi}\int_0^{2\pi} d\sigma\, \mathrm{sn}^2(\omega(\sigma+\pi/2))\right)\Psi\big|_{\mathbb{X}^{\rm ref}(0)}\,. \qquad \text{(E.10)}
\end{aligned}$$

Combining (E.9) and (E.10) with (E.4), we see that the dependence on τ_i and τ_f precisely cancel and we get

$$\Psi\big|_{\mathbb{X}^{\rm ref}(\tau_i)}\Psi\big|_{\mathbb{X}^{\rm ref}(-\tau_f)}\exp\left(-S[X]\big|_{\mathbb{X}^{\rm ref}}\right) = \left(\Psi\big|_{\mathbb{X}^{\rm ref}(0)}\right)^2\,, \qquad \text{(E.11)}$$

as announced in (4.38).

We now discuss what this result implies concerning the nature of the additional contribution of the form $\log\langle X(\sigma)|\phi_i\rangle$ at τ_i and τ_f discussed in Appendix D. Such a contribution must be considered since the action is evaluated with the string coordinates $X_\mu(\sigma)$ while the wave functions are computed in terms of the action-angle variables. First note that since the saddle point configuration is characterized by the values of the action-angle variables, the expression $\log\langle X(\sigma)|\phi_i\rangle$ can be viewed as a function of S_i and ϕ_i. In fact since S_i are constant for our solution we need to consider only the dependence on the angle variables ϕ_0 and ϕ_∞. We then recall the fact that the values of such angle variables can be shifted by arbitrary parameters a and a' as in (4.28), (4.34) and (4.35). Nevertheless the final result for the two point function given in (4.38) is independent of such parameters. This implies that the contribution of the term $\log\langle X(\sigma)|\phi_i\rangle$ for the GKP solution does not depend on ϕ_0 and ϕ_∞. Such a constant contribution can then be absorbed in the normalization of the wave function and hence does not affect the result for the two point function.

F: Evaluation of $\langle \hat{i}^L_\pm\,, \hat{j}^L_\pm\rangle$

In this appendix, we give some details of the evaluation of $\langle \hat{i}^L_\pm\,, \hat{j}^L_\pm\rangle$. As outlined in subsection 5.2, we first calculate the invariant products involving only the small solutions, namely $\langle 1^L_+, 2^L_-\rangle$, $\langle 2^L_-, 3^L_+\rangle$ and $\langle 3^L_+, 1^L_+\rangle$, in the same manner as the corresponding quantities

in the R sector. Since the prefactors $\exp\left(i(\xi^{-1}-\xi)\kappa_i \log\epsilon_i\right)$ in (5.46) are simply unity for $\xi = 1$, these products are obtained as

$$\log\langle 1_+^L, 2_-^L\rangle = \hat{h}(\kappa_1) + \hat{h}(\kappa_2) - \hat{h}(\frac{\kappa_1+\kappa_2+\kappa_3}{2}) - \hat{h}(\frac{\kappa_1+\kappa_2-\kappa_3}{2})\,, \tag{F.1}$$

$$\log\langle 2_-^L, 3_+^L\rangle = \hat{h}(\kappa_2) + \hat{h}(\kappa_3) - \hat{h}(\frac{\kappa_1+\kappa_2+\kappa_3}{2}) - \hat{h}(\frac{-\kappa_1+\kappa_2+\kappa_3}{2})\,, \tag{F.2}$$

$$\log\langle 3_+^L, 1_+^L\rangle = \hat{h}(\kappa_3) + \hat{h}(\kappa_1) - \hat{h}(\frac{\kappa_1+\kappa_2+\kappa_3}{2}) - \hat{h}(\frac{\kappa_1+\kappa_2-\kappa_3}{2})\,. \tag{F.3}$$

Here the function $\hat{h}(x)$ is obtained from the function $h(x,\xi)$ defined by

$$h(x,\xi) \equiv -\frac{1}{\pi i}\int_0^\infty d\xi' \frac{1}{\xi'^2-\xi^2}\log\left(1-\exp\left(-2\pi x({\xi'}^{-1}+\xi')\right)\right)\,, \tag{F.4}$$

by the limiting procedure $\lim_{\epsilon\to+0} h(x, 1+i\epsilon)$. Although $h(x,\xi)$ is a complicated function for general values of x and ξ, it simplifies enormously in the limit above as

$$\begin{aligned}\hat{h}(x) = &-\frac{1}{2\pi i}\left[\int_0^\infty d\xi' \frac{1}{\xi'^2-(1+i\epsilon)}\log\left(1-\exp\left(-2\pi x({\xi'}^{-1}+\xi')\right)\right)\right.\\ &\left.+\int_0^\infty d{\xi'}^{-1}\frac{1}{\xi'^{-2}-(1+i\epsilon)}\log\left(1-\exp\left(-2\pi x({\xi'}^{-1}+\xi')\right)\right)\right]\\ =&-\frac{1}{2\pi i}\int_0^\infty d\xi'\left(\frac{1}{\xi'^2-(1+i\epsilon)}-\frac{1}{\xi'^2-(1-i\epsilon)}\right)\log\left(1-e^{-2\pi x({\xi'}^{-1}+\xi')}\right)\,.\end{aligned} \tag{F.5}$$

Since the first factor in the integrand is recognized as $\delta({\xi'}^2-1)$, $\hat{h}(x)$ finally reduces to a simple function given by

$$\hat{h}(x) = -\frac{1}{2}\log\left(1-e^{-4\pi x}\right)\,. \tag{F.6}$$

Next we recall that in [51] the invariant products of $i_\pm$ are determined up to three constants, c, A_2 and A_3, which are connected to the normalization of $i_\pm$. Specifically, they are related to the invariant products obtained above as (see equations (3.69), (3.72) and (3.65) of [51])

$$\langle 1_+^L, 2_-^L\rangle = \frac{1}{2A_2\sinh 2\pi\kappa_2}\,, \tag{F.7}$$

$$\langle 2_-^L, 3_+^L\rangle = \frac{A_3\left(e^{2\pi(\kappa_1-\kappa_3)}-e^{2\pi\kappa_2}\right)}{2A_2\sinh 2\pi\kappa_2} = -\frac{A_3\sinh\left(\pi(-\kappa_1+\kappa_2+\kappa_3)\right)}{A_2\sinh 2\pi\kappa_2}e^{\pi(\kappa_1+\kappa_2-\kappa_3)}\,, \tag{F.8}$$

$$\langle 3_+^L, 1_+^L\rangle = -cA_3\,. \tag{F.9}$$

From these relations, we get

$$A_2 = \frac{1}{2\sinh 2\pi\kappa_2 \langle 1_+ \,, 2_- \rangle} \,, \tag{F.10}$$

$$A_3 = -\frac{\langle 2_- \,, 3_+ \rangle}{\langle 1_+ \,, 2_- \rangle} \frac{e^{-\pi(\kappa_1+\kappa_2-\kappa_3)}}{2\sinh\left(\pi(-\kappa_1+\kappa_2+\kappa_3)\right)} \,, \tag{F.11}$$

$$c = 2\frac{\langle 3_+ \,, 1_+ \rangle \langle 1_+ \,, 2_+ \rangle}{\langle 2_- \,, 3_+ \rangle} e^{\pi(\kappa_1+\kappa_2-\kappa_3)} \sinh\left(\pi(-\kappa_1+\kappa_2+\kappa_3)\right) \,. \tag{F.12}$$

With these expressions for c , A_2 and A_3, we can compute the remaining invariant products given in [51]. For instance, the products involving big solutions around z_1 and z_2, namely $\langle 1_-^L, 2_+^L \rangle$, $\langle 1_+^L, 2_+^L \rangle$ and $\langle 1_-^L, 2_-^L \rangle$, are obtained as

$$\langle 1_-^L, 2_+^L \rangle = -\frac{1}{\langle 1_+ \,, 2_- \rangle} \frac{\sinh\left(\pi(\kappa_1+\kappa_2+\kappa_3)\right) \sinh\left(\pi(\kappa_1+\kappa_2-\kappa_3)\right)}{\sinh 2\pi\kappa_1 \sinh 2\pi\kappa_2} \,, \tag{F.13}$$

$$\langle 1_+^L, 2_+^L \rangle = \frac{\langle 3_+ \,, 1_+ \rangle}{\langle 2_- \,, 3_+ \rangle} \frac{\sinh\left(\pi(-\kappa_1+\kappa_2+\kappa_3)\right)}{\sinh 2\pi\kappa_2} e^{\pi(\kappa_1+\kappa_2-\kappa_3)} \,, \tag{F.14}$$

$$\langle 1_-^L, 2_-^L \rangle = \frac{\langle 2_- \,, 3_+ \rangle}{\langle 3_+ \,, 1_+ \rangle} \frac{\sinh\left(\pi(\kappa_1-\kappa_2+\kappa_3)\right)}{\sinh 2\pi\kappa_1} e^{-\pi(\kappa_1+\kappa_2-\kappa_3)} \,. \tag{F.15}$$

Combining these results, the invariant products between $\hat{i}_\pm^L \equiv (\pm i_+^L + i_-^L)/\sqrt{2}$ are given by

$$\langle \hat{1}_+^L, \hat{2}_-^L \rangle = \tilde{c}\sqrt{\frac{\sinh\left(\pi(\kappa_1+\kappa_2-\kappa_3)\right) \sinh\left(\pi(\kappa_1+\kappa_2+\kappa_3)\right)}{\sinh(2\pi\kappa_1)\sinh(2\pi\kappa_2)}} \,, \tag{F.16}$$

$$\langle \hat{2}_-^L, \hat{3}_+^L \rangle = \tilde{c}\sqrt{\frac{\sinh\left(\pi(-\kappa_1+\kappa_2+\kappa_3)\right) \sinh\left(\pi(\kappa_1+\kappa_2+\kappa_3)\right)}{\sinh(2\pi\kappa_2)\sinh(2\pi\kappa_3)}} \,, \tag{F.17}$$

$$\langle \hat{3}_+^L, \hat{1}_+^L \rangle = \tilde{c}\sqrt{\frac{\sinh\left(\pi(\kappa_1-\kappa_2+\kappa_3)\right) \sinh\left(\pi(\kappa_1+\kappa_2+\kappa_3)\right)}{\sinh(2\pi\kappa_3)\sinh(2\pi\kappa_1)}} \,, \tag{F.18}$$

where $\tilde{c}$ is the expression given by

$$\tilde{c} = 1 - \sqrt{\frac{\sinh(\pi(-\kappa_1+\kappa_2+\kappa_3))\sinh(\pi(\kappa_1-\kappa_2+\kappa_3))\sinh(\pi(\kappa_1+\kappa_2-\kappa_3))}{\sinh(\pi(\kappa_1+\kappa_2+\kappa_3))}} \,. \tag{F.19}$$

G: Two point function for a particle in Euclidean AdS_3

In section 4, we developed, through somewhat abstract reasoning based on integrability, a powerful method of computing the two point functions, which are applicable even to the cases where the action-angle variables are difficult to construct explicitly. It is instructive, however, to check the method by applying it to a non-trivial system for which the action-angle variables *can* be obtained analytically.

One such system is a particle in Euclidean AdS_3. If we employ the global coordinates (θ, ϕ, ρ), related to the embedding coordinates X_μ by

$$X_{-1} = \cosh\rho\cosh\phi\,, \qquad X_4 = \cosh\rho\sinh\phi\,, \tag{G.1}$$

$$X_1 = \sinh\rho\cos\theta\,, \qquad X_2 = \sinh\rho\sin\theta\,, \tag{G.2}$$

the action takes the form

$$S = \int d\tau\,\frac{1}{2}\left(-\dot{\rho}^2 - \cosh^2\rho\,\dot{\phi}^2 + \sinh^2\rho\,\dot{\theta}^2\right)\,. \tag{G.3}$$

The action variables corresponding to the motions of θ, ϕ and ρ are defined as

$$J_\theta = \frac{1}{2\pi}\oint p_\theta d\theta\,, \quad J_\phi = \frac{1}{2\pi}\oint p_\phi d\phi\,, \quad J_\rho = \frac{1}{2\pi}\oint p_\rho d\rho\,. \tag{G.4}$$

Since the motions of ϕ and ρ are actually not periodic, we need to perform the analytic continuations $\tilde{\rho} = i\rho$ and $\tilde{\phi} = i\phi$ to define the periodic integrals above. The angle variables $(\varphi_\theta, \varphi_\phi, \varphi_\rho)$ conjugate to $(J_\theta, J_\phi, J_\rho)$ can be constructed as

$$\frac{\partial F}{\partial J_\theta} = \varphi_\theta\,, \quad \frac{\partial F}{\partial J_\phi} = \varphi_\phi\,, \quad \frac{\partial F}{\partial J_\rho} = \varphi_\rho\,, \tag{G.5}$$

where $F \equiv \int d\theta\, p_\theta + \int d\phi\, p_\phi + \int d\rho\, p_\rho$ is the generating function. To compute these integrals, one must express the momenta $(p_\theta, p_\phi, p_\rho)$ in terms of the action variables and the coordinates (θ, ϕ, ρ). The general form of the angle variables obtained through this procedure are rather complicated, but for $J_\theta = J_\rho = 0$ they take the following simple form:

$$\varphi_\phi = -\phi + \ln\cosh\rho\,, \quad \varphi_\theta = \theta + \ln\sinh\rho\,, \quad \varphi_\rho = -2i\ln\sinh\rho\,. \tag{G.6}$$

We can now evaluate these angle variables explicitly on the reference solution $\mathbb{X}^{\text{ref}}$, given by

$$\mathbb{X}^{\text{ref}} = \begin{pmatrix} e^{-\kappa\tau} & 0 \\ 0 & e^{\kappa\tau} \end{pmatrix}\,, \tag{G.7}$$

and on the transformed solution $\mathbb{X} \equiv V_L\mathbb{X}^{\text{ref}}V_R$, with $V_{L,R}$ given in (4.27), just as in section 4. The results for the shifts are

$$\Delta\varphi_\phi^{\mathbb{X}} = -\log(aa')\,, \quad \Delta\varphi_\phi^{\tilde{\mathbb{X}}} = \log(aa'x_0\bar{x}_0)\,. \tag{G.8}$$

These shifts contribute to the two point function as

$$\exp\left(-\Delta(\Delta\varphi_\phi^{\mathbb{X}} + \Delta\varphi_\phi^{\tilde{\mathbb{X}}})\right) = \frac{1}{x_0^\Delta\bar{x}_0^\Delta}\,, \tag{G.9}$$

which reproduces the correct spacetime behavior. The remaining contributions, *i.e.* those from the action and the reference wave functions, cancel as in section 4.2 and we obtain the properly normalized two point function.

H: Exact solution describing a scattering of three spinning strings in flat space and its action-angle variables

Construction of three-pronged solutions in (the subspace of) $AdS_5 \times S^5$ is an important challenging problem. As discussed in section 3.2.4, their analytic structure is expected to be qualitatively quite different from that of the two point solutions. To give support to this observation, we present below an exact solution describing a scattering of three spinning strings in flat space and analyze its local behavior. This confirms some important structures concerning the action-angle variables.

A solution describing three interacting strings spinning in the x_1-x_2 plane is given by

$$X^\mu = -i\left(k_1^\mu \ln|z| + k_2^\mu \ln|z-1| + k_3^\mu \ln|z-\infty|\right)\,, \quad \mu \neq 1,2\,, \tag{H.1}$$

$$X = \frac{w_3}{2i}(z-\bar{z})\,, \quad \bar{X} = \frac{w_1}{2i}\left(\frac{1}{z}-\frac{1}{\bar{z}}\right) + \frac{w_2}{2i}\left(\frac{1}{z-1}-\frac{1}{\bar{z}-1}\right)\,. \tag{H.2}$$

Here, as indicated, X^μ denotes the directions other than the plane of rotation, X and $\bar{X}$ stand for $X_1 + iX_2$ and its complex conjugate respectively and the momentum vectors $\vec{k}_i$ and the parameters w_i, which are related to the spins of the prongs, must satisfy the following conservation laws and the on-shell conditions demanded by the Virasoro conditions:

$$\begin{aligned} &\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0\,, \quad w_1 + w_2 = w_3\,, \\ &(\vec{k}_1)^2 + w_1 w_3 = (\vec{k}_2)^2 + w_2 w_3 = (\vec{k}_3)^2 + (w_3)^2 = 0\,. \end{aligned} \tag{H.3}$$

Let us study its local behavior by focusing on the vicinity of the singularity $z=0$. The expansion around this point reads

$$X = \frac{w_3}{2i}(z-\bar{z})\,, \quad \bar{X} = \frac{w_1}{2i}\left(\frac{1}{z}-\frac{1}{\bar{z}}\right) - \frac{w_2}{2i}(z-\bar{z}) + O(|z|^2)\,. \tag{H.4}$$

This should be compared with the well-known two-point spinning string solution of [64] given by

$$X = \frac{w}{2i}(z-\bar{z})\,, \quad \bar{X} = \frac{w}{2i}\left(\frac{1}{z}-\frac{1}{\bar{z}}\right)\,. \tag{H.5}$$

There are two important differences to be noted. First, the Fourier coefficient in front of the structure $(z-\bar{z})$ and the one in front of $(1/z - 1/\bar{z})$ are different in (H.4), while they are the same in (H.5). Since the log of the ratio of such Fourier coefficients describes the shift of the angle variable, this means that the presence of the other vertex operator generated a shift of the angle variable in the case of the three-pronged solution. This type

of phenomenon was observed also in the calculation of the correlation functions performed in sections 4 and 5. The second feature is that for (H.4) there are an infinite number of additional Fourier modes excited in $\bar{X}$ besides $(1/z - 1/\bar{z})$. However, since there are no corresponding modes in X, these additional excitations do not contribute to the action variable, namely the spin, given by

$$S = \frac{i}{4\pi\alpha'} \int_0^{2\pi} d\sigma (X\dot{\bar{X}} - \dot{X}\bar{X}) \,. \tag{H.6}$$

This means that the infinite number of action variables corresponding to such additional Fourier modes must vanish. Therefore, the solution above embodies the general feature expected of the solution for the higher-point functions. Namely, such a solution has (possibly infinitely many) dynamical angle variables for which the conjugate action variables are zero, in addition to those associated with the action variables which are finite. This suggests that solutions for higher-point functions in Euclidean AdS_3 may be constructed also by introducing infinitely many additional degenerate cuts on the spectral curve.

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